ON THE STRUCTURE OF CERTAIN VALUED FIELDS

JUNGUK LEE AND WAN LEE

Abstract. For any two complete discrete valued fields K_1 and K_2 of mixed characteristic with perfect residue fields, we show that if each pair of n-th residue rings is isomorphic for each $n \geq 1$, then K_1 and K_2 are isometric and isomorphic. More generally, for $n_1, n_2 \geq 1$, if n_2 is large enough, then any homomorphism from the n_1-th residue ring of K_1 to the n_2-th residue ring of K_2 can be lifted to a homomorphism between the valuation rings. We can find a lower bound for n_2 depending only on K_2. Moreover, we get a functor from a category of certain principal Artinian local rings of length n to a category of certain complete discrete valuation rings of mixed characteristic with perfect residue fields, which naturally generalizes the functorial property of unramified complete discrete valuation rings. The result improves Basarab's generalization of the AKE-principle for finitely ramified henselian valued fields, which solves a question posed by Basarab, in the case of perfect residue fields.

1. Introduction

In this paper, we consider some problems on valued fields arising from the interaction of number-theoretic approaches and model-theoretic approaches. In number theory, it is well-known that the following are equivalent.

- There is an isometric isomorphism between two complete unramified discrete valued fields K_1 and K_2 of mixed characteristic $(0, p)$ with perfect residue fields.
- There is an isomorphism between residue fields of K_1 and K_2.

In model theory, as a counterpart, there is a principle called Ax-Kochen-Ershov-principle(briefly, *AKE-principle*) which states the following are equivalent.

- Two absolutely unramified henselian valued fields K_1 and K_2 of the same type of mixed characteristic $(0, p)$ with value groups as $\mathbb{Z}$-groups are elementarily equivalent.
- Residue fields of K_1 and K_2 are elementarily equivalent.

We introduce more elementary classes of valued fields satisfying the AKE-principle:
a) Algebraically closed valued fields by Robinson in [28].
b) Henselian fields with residue fields of characteristic 0 by Ax and Kochen in [4] and independently by Ershov in [14].
c) p-adically closed fields by Ax and Kochen in [4] and independently by Ershov in [14].
d) Algebraically maximal Kaplansky fields by Ershov in [15] and independently by Ziegler in [31].
e) Tame fields of equal characteristic by Kuhlmann in [22].
f) Separably tame fields of equal characteristic by Kuhlmann and Pal in [23].
Some elementary classes of valued fields with additional structures are also known to satisfy the AKE-principle.

g) The ring of Witt vectors after adding a predicate for a unique multiplicative set of representatives for the residue field by van den Dries in [12].
h) Some valued difference fields by Bélair, Macintyre, and Scanlon in [9], by Azgin and van den Dries in [1], and by Pal in [26].

Most of all, in this paper, we are interested in finitely ramified valued fields. Prestel and Roquette in [27] considered the class of $\wp$-closed fields which are finite extensions of p-closed fields so that the residue fields are finite. They showed that the theory of $\wp$-closed fields of a fixed p-rank is model complete. Basarab in [6] extended this result and generalized the AKE-principle for the case of finitely ramified valued fields. Actually, he showed that for any two finitely ramified henselian valued fields of mixed characteristic, they are elementarily equivalent if and only if their value groups are elementarily equivalent, and their n-th residue rings are elementarily equivalent for each $n \geq 1$, where the n-th residue ring is the quotient of the valuation ring by the n-th power of the maximal ideal. And the theory of a finitely ramified henselian valued field is model complete if and only if each theory of its n-th residue ring and its value group are model complete. Motivated from the relation between number theory and model theory for the unramified case, we ask whether there is a number-theoretic part which corresponds to Basarab's result on the AKE-principle:

Question 1.1. *Are two finitely ramified complete discrete valued fields K_1 and K_2 of mixed characteristic with perfect residue fields isomorphic if the n-th residue rings of K_1 and K_2 are isomorphic for each $n \geq 1$?*

We report some known necessary and sufficient conditions for certain valued fields to be isomorphic. For a p-valued field (K, ν) and any two $\wp$-closed fields (L_1, ν) and (L_2, ν) of the same p-rank as (K, ν), Prestel and Roquette in [27] showed that L_1 and L_2 are K-isomorphic as valued fields if and only if the n-th powers of L_1 and L_2 contained in K are the same for each n. Basarab and Kuhlmann in [7] introduced some structure called the mixed δ-structure for each δ in the value group of a valued field. By using these mixed structures, for any two henselian algebraic extensions (L_1, ν) and (L_2, ν) of a given valued field (K, ν), they gave a criterion for (L_1, ν) and (L_2, ν) to be K-isomorphic as valued fields under a certain condition with respect to tame extensions.

We return to the unramified case. The previous equivalence for the unramified case in number theory is a corollary of the following well-known theorem([29]).

- For any perfect field k of characteristic p, there exists a unique unramified complete discrete valuation ring R, called the ring of Witt vectors of k, of characteristic 0 which has k as its residue field.
- For any two unramified complete discrete valuation rings R_1 and R_2 of mixed characteristic with perfect residue fields k_1 and k_2 respectively, suppose that there is a homomorphism $\phi : k_1 \longrightarrow k_2$. Then there is a unique lifting homomorphism $g : R_1 \longrightarrow R_2$ such that g induces ϕ.

In categorical setting, the theorem above is equivalent to the following statement.

- Let $\mathcal{C}_p$ be a category of complete unramified discrete valuation rings of mixed characteristic $(0, p)$ with perfect residue fields and $\mathcal{R}_p$ a category of perfect fields of characteristic p. Then $\mathcal{C}_p$ is equivalent to $\mathcal{R}_p$. More precisely, there is a functor $\mathrm{L}' : \mathcal{R}_p \to \mathcal{C}_p$ which satisfies:

– $\mathrm{Pr} \circ \mathrm{L}'$ is equivalent to the identity functor $\mathrm{Id}_{\mathcal{R}_p}$ where $\mathrm{Pr} : \mathcal{C}_p \longrightarrow \mathcal{R}_p$ is the natural projection functor.
– $\mathrm{L}' \circ \mathrm{Pr}$ is equivalent to $\mathrm{Id}_{\mathcal{C}_p}$.

Based on Question 1.1 and the statements above, we raise generalized questions.

Question 1.2. *(1) For a principal Artinian local ring $\overline{R}$ of length n with certain conditions, is there a unique complete discrete valuation ring R which has $\overline{R}$ as its residue ring ?*

(2) For any two finitely ramified complete discrete valuation rings R_1 and R_2 of mixed characteristic with perfect residue fields, let R_{1,n_1} and R_{2,n_2} be the n_1-th residue ring of R_1 and the n_2-th residue ring of R_2 respectively. Under certain conditions on n_1 and n_2, given a homomorphism $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$, is there a unique lifting homomorphism $g : R_1 \longrightarrow R_2$ such that g induces ϕ ?

Question 1.3. *Let $\mathcal{C}_{p,e}$ be a category of complete discrete valuation rings of mixed characteristic $(0, p)$ with perfect residue fields and absolute ramification index e. Let $\mathcal{R}^n_{p,e}$ be a category of principal Artinian local rings of length n with certain conditions. Let $\mathrm{Pr}_n : \mathcal{C}_{p,e} \longrightarrow \mathcal{R}^n_{p,e}$ be the natural projection functor. Is there a functor $\mathrm{L} : \mathcal{R}^n_{p,e} \longrightarrow \mathcal{C}_{p,e}$ which satisfies:*

- $\mathrm{Pr}_n \circ \mathrm{L}$ *is equivalent to* $\mathrm{Id}_{\mathcal{R}^n_{p,e}}$.
- $\mathrm{L} \circ \mathrm{Pr}_n$ *is equivalent to* $\mathrm{Id}_{\mathcal{C}_{p,e}}$.

Question 1.2.(2) is not true in general, that is, there is a homomorphism $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$ such that any homomorphism from R_1 into R_2 does not induce ϕ. In this paper, the main result shows that for sufficiently large n_2, if there is a given homomorphism $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$, then there is a homomorphism $g : R_1 \longrightarrow R_2$ rather naturally related with ϕ. In the beginning of Section 2, we show that Question 1.1 is true in a special case of local fields. The main ingredient in the proof is to use the compactness of the valuation rings of local fields. In order to extend the result to the case of infinite perfect residue fields, we need the Witt subring. Since valuation rings are not compact in general, we use Krasner's lemma instead. More precisely, the main result shows that for sufficiently large n_2, if there is a given homomorphism $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$, then there is a homomorphism $\mathrm{L}(\phi) : R_1 \longrightarrow R_2$ satisfying a lifting property similar to that of the unramified case. Even though the construction of $\mathrm{L}(\phi)$ depends on the choice of uniformizer, it turns out that $\mathrm{L}(\phi)$ does not depend on the choice of uniformizer. Moreover, when ϕ is an isomorphism, so is $\mathrm{L}(\phi)$. This provides an answer for Question 1.1. We define $\mathrm{L}(\phi)$ as the lifting of ϕ even though $\mathrm{L}(\phi)$ does not induce ϕ. The lifting map L provides an answer for Question 1.2.(2) and Question 1.2.(1) where the latter follows from L and the Cohen structure theorem for complete local ring([19]).

In Section 3, we concentrate on Question 1.3. By using the fact that the definition of the lifting map L is independent of the choice of uniformizer, we can show that L is compatible with composition of homomorphisms between residue rings. More precisely, $\mathrm{L}(\phi_2 \circ \phi_1) = \mathrm{L}(\phi_2) \circ \mathrm{L}(\phi_1)$ for any $\phi_1 : R_{1,n_1} \longrightarrow R_{2,n_2}$ and $\phi_2 : R_{2,n_2} \longrightarrow R_{3,n_3}$. This defines a functor $\mathrm{L} : \mathcal{R}^n_{p,e} \longrightarrow \mathcal{C}_{p,e}$ for sufficiently large n. We prove that a lower bound for n depends only on the ramification index e and the prime number p. Even though L does not give an equivalence between $\mathcal{R}^n_{p,e}$ and $\mathcal{C}_{p,e}$, it turns out that L satisfies a similar functorial property to $\mathrm{L}' : \mathcal{R}_p \to \mathcal{C}_p$. This provides an answer for Question 1.3.

We define the lifting number for $\mathcal{C}_{p,e}$ as the least number n such that there is a lifting functor $\mathrm{L} : \mathcal{R}^n_{p,e} \longrightarrow \mathcal{C}_{p,e}$. For the tamely ramified case, we prove that the lifting number for $\mathcal{C}_{p,e}$ is $e+1$ when $e \geq 2$. For the wildly ramified case, we have that the lifting number for $\mathcal{C}_{p,e}$ is at least $e+1$. Finally we conclude that the lifting number for $\mathcal{C}_{p,e}$ is either $=1$ or ≥ 3 for any case. We note that the lifting number for $\mathcal{C}_{p,e}$ is 1 if and only if $e = 1$.

In [6], Basarab posed the following question:

Question 1.4. *Given a finitely ramified henselian valued field K of ramification index $e \geq 2$, is there a finite integer $N' \geq 1$ depending on K such that any other finitely ramified henselian valued field of the same ramification index e is elementarily equivalent to K if and only if their N'-th residue rings are elementarily equivalent and their value groups are elementarily equivalent?*

In Section 4, for given valued fields, each of whose value groups has a least positive element, we reduce the problem determining elementary equivalence between them to the problem determining whether certain complete discrete valued fields related with them are isomorphic. Using results in Section 2, we improve Basarab's result on the AKE-principle which gives a positive answer for Question 1.4 when the residue fields are perfect.

Given a finitely ramified henselian valued field K, Basarab([6]) denoted the minimal number N' which satisfies the equivalence in Question 1.4 by $\lambda(T)$ for a complete theory T of K. $\lambda(T)$ can be 1 even when K is not unramified. Under certain conditions, we calculate $\lambda(T)$ explicitly for the tame case and get a lower bound of $\lambda(T)$ for the wild case. As a special case, we conclude that $\lambda(T)$ is 1 or $e+1$ if $p \nmid e$, and $\lambda(T) \geq e+1$ if $p|e$ when K is a finitely ramified henselian subfield of $\mathbb{C}_p$ with ramification index e.

We introduce basic notations and terminologies which will be used in this paper. We denote a valued field by a tuple $(K, R, \mathfrak{m}, \nu, k, \Gamma)$ consisting of the following data : K is the underlying field, R is the valuation ring, $\mathfrak{m}$ is the maximal ideal of R, ν is the valuation map, k is the residue field, and Γ is the value group. Hereafter, the full tuple $(K, R, \mathfrak{m}, \nu, k, \Gamma)$ will be abbreviated in accordance with the situational need for the components.

Definition 1.5. *Let (K, ν, k, Γ) be a valued field of characteristic zero. We say (K, ν) is* absolutely unramified *if $char(k) = 0$, or $char(k) = p$ and $\nu(p)$ is the minimal positive element in Γ for $p > 0$. We say (K, ν) is* absolutely ramified *if it is not absolutely unramified.*

Definition 1.6. *Let (K, ν, k, Γ, R) be a valued field whose residue field has prime characteristic p.*

1. *We say (K, ν, k, Γ, R) is* absolutely finitely ramified *if the set $\{\gamma \in \Gamma | \ 0 < \gamma \leq \nu(p)\}$ is finite. The cardinality of $\{\gamma \in \Gamma | \ 0 < \gamma \leq \nu(p)\}$ is called* the absolute ramification index *of (K, ν), denoted by $e(K, \nu)$ or $e(R)$. If K or ν is clear from context, we write $e(K)$ or e for $e(K, \nu)$. For $x \in R$, we write $e_\nu(x) := |\{\gamma \in \Gamma | \ 0 < \gamma \leq \nu(x)\}|$. If there is no confusion, we write $e(x)$ for $e_\nu(x)$*
2. *Let (K, ν, k, Γ, R) be finitely ramified. If p does not divide $e_\nu(p)$, we say (K, ν) is* absolutely tamely ramified. *Otherwise, we say (K, ν) is* absolutely wildly ramified.

Note that if a valued field of mixed characteristic has the absolute finite ramification index, then its value group has the minimum positive element.

Definition 1.7. *Let* (K_1, ν_1) *and* (K_2, ν_2) *be valued fields. Let* R_1 *and* R_2 *be subrings of* K_1 *and* K_2 *respectively. Let* $f : R_1 \to R_2$ *be a injective ring homomorphism. We say* f *is an* isometry *if for* $a, b \in R_1$,

$$\nu_1(a) > \nu_1(b) \Leftrightarrow \nu_2(f(a)) > \nu_2(f(b)).$$

Definition 1.8. *For a local ring* R *with maximal ideal* $\mathfrak{m}$*, we denote* $R/\mathfrak{m}^n$ *by* R_n*, and we call* R_n *the* n-th residue ring *of* R*. In particular,* R_1 *is the residue field of* R*.*

For each $m > n$, let $\mathrm{pr}_n : R \to R_n$ and $\mathrm{pr}_n^m : R_m \to R_n$ be the canonical projection maps respectively. For R-algebras S_1 and S_2, we denote the set of R-algebra homomorphisms from S_1 to S_2 by $\mathrm{Hom}_R(S_1, S_2)$, and we briefly write $\mathrm{Hom}(S_1, S_2)$ for $\mathrm{Hom}_{\mathbb{Z}}(S_1, S_2)$. We denote the set of R-algebra isomorphisms by $\mathrm{Iso}_R(S_1, S_2)$, and we write $\mathrm{Iso}(S_1, S_2)$ for $\mathrm{Iso}_{\mathbb{Z}}(S_1, S_2)$. We write $\mathrm{Iso}(R)$ for $\mathrm{Iso}(R, R)$. We denote a primitive n-th root of unity by ζ_n.

2. Lifting homomorphisms

We start from the following proposition.

Proposition 2.1. *Let* K_1 *and* K_2 *be finite extensions of* $\mathbb{Q}_p$ *for some prime* p*. Let* $R_{1,n}$ *and* $R_{2,n}$ *be the* n*-th residue rings of* K_1 *and* K_2 *respectively. Suppose that there is an isomorphism* $\iota_n : R_{1,n} \to R_{2,n}$ *for each* $n > 0$*. Then there is an isomorphism* $\iota : K_1 \to K_2$ *over* $\mathbb{Q}_p$*.*

Proof. (1) First method: Let $\mathrm{Iso}(R_n)$ be the set of isomorphisms from $R_{1,n}$ onto $R_{2,n}$, and $\xi_{n+1,n}$ be the natural reduction map from $\mathrm{Iso}(R_{n+1})$ to $\mathrm{Iso}(R_n)$. Then $\{\mathrm{Iso}(R_n), \xi_{n+1,n}\}$ forms an inverse system. Since each residue ring $R_{i,n}$ is finite, $\mathrm{Iso}(R_n)$ is finite,in particular compact for each n. By the theory of topological algebra, $\varprojlim \mathrm{Iso}(R_n)$ is not empty([25]). This shows there exists an isomorphism $\iota : K_1 \longrightarrow K_2$. ι is defined over $\mathbb{Q}_p$ since all elements of $\mathrm{Iso}(R_n)$ are continuous.

(2) Second method: Let R_1 and R_2 be valuation rings of K_1 and K_2 respectively. Take an element a in R_1 satisfying $K_1 = \mathbb{Q}_p(a)$. Let f be the monic irreducible polynomial of a over $\mathbb{Z}_p$. Consider a sequence $(a'_n \in R_2)_{n \geq 1}$ such that $\mathrm{pr}_{2,n}(a'_n) = \iota_n(\mathrm{pr}_{1,n}(a))$ where $\mathrm{pr}_{i,n}$ denotes a n-th natural projection from R_i to $R_{i,n}$. We note that each ι_n is an $\mathbb{Z}_p$-algebra isomorphism since ι_n is continuous. Since $f(a) = 0$, $f(\iota_n(\mathrm{pr}_{1,n}(a))) = \iota_n(f(\mathrm{pr}_{1,n}(a))) = \iota_n(\mathrm{pr}_{1,n}(f(a))) = 0$ in $R_{2,n}$. First equality follows from that fact that ι_n is an $\mathbb{Z}_p$-algebra homomorphism. Hence, $f(\mathrm{pr}_{2,n}(a'_n)) = \mathrm{pr}_{2,n}(f(a'_n)) = 0$ in $R_{2,n}$, that is, $f(a'_n) \in \mathfrak{m}_2^n$ where $\mathfrak{m}_2$ is the maximal ideal of R_2. Since R_2 is compact, there is a subsequence (a'_{n_i}) which converges to $a' \in R_2$, and since f is continuous, $f(a') = \lim_{n_i \to \infty} f(a'_{n_i}) = 0$ in R_2. Thus K_2 contains a zero a' of f. Therefore, there is an injection $\iota : K_1 \to K_2,\ a \mapsto a'$ over $\mathbb{Q}_p$, and hence, we obtain an inequality $[K_1 : \mathbb{Q}_p] \leq [K_2 : \mathbb{Q}_p]$ between the field extension degrees. Similarly, one can show $[K_1 : \mathbb{Q}_p] \geq [K_2 : \mathbb{Q}_p]$. Hence, ι is an isomorphism over $\mathbb{Q}_p$.

□

Since the proof of the fact that the inverse limit of $\mathrm{Iso}(R_n)$ is not empty uses Zorn's lemma, we can only prove the existence of an isomorphism in the first method. The second method does not use Zorn's lemma and the given isomorphism is more easier to construct. But both methods use the fact that the homomorphisms are defined over $\mathbb{Q}_p$ or $\mathbb{Z}_p$ crucially. For the case of infinite perfect residue fields, we need the absolutely unramified discrete valuation rings called the ring of Witt vectors. By Krasner's lemma, it suffices to consider a single n-th residue ring for sufficiently large n.

The following theorem is well-known.

Theorem 2.2. *(1) Let k be a perfect field of characteristic p. Then there exists a complete discrete valuation ring of characteristic 0 which is absolutely unramified and has k as its residue field. Such a ring is unique up to isomorphism. This unique ring is called the ring of Witt vectors of k, denoted by $W(k)$.*

(2) Let R_1 and R_2 be complete discrete valuation rings of mixed characteristic with perfect residue fields k_1 and k_2 respectively. Suppose R_1 is absolutely unramified. Then for every homomorphism $\phi : k_1 \longrightarrow k_2$, there exists a unique homomorphism $g : R_1 \longrightarrow R_2$ making the following diagram commutative:

$$\begin{array}{ccc} R_1 & \xrightarrow{\ g\ } & R_2 \\ {\scriptstyle \mathrm{pr}_{1,1}}\big\downarrow & & {\scriptstyle \mathrm{pr}_{2,1}}\big\downarrow \\ k_1 & \xrightarrow{\ \phi\ } & k_2 \end{array}$$

Proof. Chapter 2, section 5 of [29]. □

Before stating main theorems, we need some lemmas.

Lemma 2.3. *Let R be a complete discrete valuation ring of characteristic 0 with perfect residue field k of characteristic p and corresponding valuation ν. Then $W(k)$ can be embedded as a subring of R and R is a free $W(k)$-module of rank $\nu(p)$. Moreover, $R = W(k)[\pi]$ where π is a uniformizer of R.*

Proof. Chapter 2, Section 5 of [29] □

Lemma 2.4. *Let A be a ring that is Hausdorff and complete for a topology defined by a decreasing sequence $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \ldots$ of ideals such that $\mathfrak{a}_n \cdot \mathfrak{a}_m \subset \mathfrak{a}_{n+m}$. Assume that the residue ring $A_1 = A/\mathfrak{a}_1$ is a perfect field of characteristic p. Then:*

(1) There exists one and only one system of representatives $h : A_1 \longrightarrow A$ which commutes with p-th powers: $h(\lambda^p) = h(\lambda)^p$. This system of representatives is called the set of Teichmüller representatives.

(2) In order that $a \in A$ belong to $S = h(A_1)$, it is necessary and sufficient that a be a p^n-th power for all $n \geq 0$.

(3) This system of representatives is multiplicative which means

$$h(\lambda\mu) = h(\lambda)h(\mu)$$

for all $\lambda, \mu \in A_1$.

(4) S contains 0 and 1.

(5) $S \setminus \{0\}$ is a subgroup of the unit group of A.

Proof. (1)(2)(3): Chapter 2, Section 4 of [29]

(4): 0 and 1 satisfy (2).

(5): (3) and (4) show that $S \setminus \{0\}$ is a subgroup of the unit group of A. □

Lemma 2.5. *Let R_1 and R_2 be discrete valuation rings of characteristic* 0 *with residue characteristic p. Let $\mathfrak{m}_i$ be the maximal ideal of R_i generated by π_i and ν_i corresponding discrete valuation of R_i for $i = 1, 2$. Suppose there is a homomorphism $\iota : R_1 \longrightarrow R_{2,n}$. If $n > a\nu_2(p)$ for some real number $a \geq 1$, kernel of ι is equal to $\mathfrak{m}_1^m$ for some $m > a\nu_1(p)$.*

Proof. Let $\overline{\mathfrak{m}}_2 = \mathfrak{m}_2/\mathfrak{m}_2^n$ be the maximal of $R_{2,n}$. If we write $\iota(\pi_1)R_{2,n} = \overline{\mathfrak{m}}_2^x$,

$$\overline{\mathfrak{m}}_2^{\nu_2(p)} = pR_{2,n} = \iota(p)R_{2,n}$$
$$= \iota\left(\pi_1^{\nu_1(p)}\right)R_{2,n} = \overline{\mathfrak{m}}_2^{x\nu_1(p)}.$$

In particular $x = \nu_2(p)/\nu_1(p)$ and $\nu_2(p)/\nu_1(p)$ is a positive integer. Suppose

$$\iota(\pi_1^m)R_{2,n} = \overline{\mathfrak{m}}_2^{\frac{m\nu_2(p)}{\nu_1(p)}} = 0$$

in $R_{2,n}$ for some m. Then we obtain

$$\frac{m\nu_2(p)}{\nu_1(p)} \geq n > a\nu_2(p),$$

and hence $m > a\nu_1(p)$. □

For any field L, L^{alg} denotes a fixed algebraic closure of L. Let (L, ν) be a valued field whose value group is contained in $\mathbb{R}$. If L is of characteristic 0 and of residue characteristic p, we define a normalized valuation $\overline{\nu}$ on L of ν by the property $\overline{\nu}(p) = 1$, that is, $\nu(p)\overline{\nu} = \nu$. We denote an extended valuation of $\overline{\nu}$ on L^{alg} by $\widetilde{\nu}$. When L is henselian, $\widetilde{\nu}$ is unique.

Lemma 2.6. *Let (K_1, ν_1) and (K_2, ν_2) be valued fields whose value groups are contained in $\mathbb{R}$. Let $f : K_1 \longrightarrow K_2$ be an isometric homomorphism. Suppose K_1 is henselian. Let $\widetilde{f} : K_1^{alg} \longrightarrow K_2^{alg}$ be an extended homomorphism of f. Then $\widetilde{f}$ is an isometry.*

Proof. There are two valuations on $\widetilde{f}(K_1^{alg})$, $\widetilde{\nu_1} \circ \widetilde{f}^{-1}$ and $\widetilde{\nu_2}|_{\widetilde{f}(K_1^{alg})}$ where $\widetilde{\nu_2}|_{\widetilde{f}(K_1^{alg})}$ is the restriction of $\widetilde{\nu_2}$ to $\widetilde{f}(K_1^{alg})$. Since f is an isometry, the restrictions of $\widetilde{\nu_1} \circ \widetilde{f}^{-1}$ and $\widetilde{\nu_2}|_{\widetilde{f}(K_1^{alg})}$ to $f(K_1)$ are equivalent, in fact, they are equal since $(\widetilde{\nu_1} \circ \widetilde{f}^{-1})(p) = \widetilde{\nu_2}|_{\widetilde{f}(K_1^{alg})}(p) = 1$. Since K_1 is henselian, $f(K_1)$ is Henselian. Hence, $\widetilde{\nu_1} \circ \widetilde{f}^{-1}$ is equal to $\widetilde{\nu_2}|_{\widetilde{f}(K_1^{alg})}$ by the henselian property. This shows that $\widetilde{f}$ is an isometry. □

Let R be a complete discrete valuation ring of mixed characteristic with perfect residue field. Let π be a uniformizer of R and ν corresponding valuation of R. Let L and K be the fraction fields of R and $W(k)$ respectively. We denote the maximal number

$$\max\left\{\widetilde{\nu}\big(\pi - \sigma(\pi)\big) : \sigma \in \mathrm{Hom}_K(L, L^{alg}), \sigma(\pi) \neq \pi\right\}$$

by $M(R)_\pi$ or $M(L)_\pi$.

Definition 2.7. *Let R_1 and R_2 be complete discrete valuation rings of characteristic* 0 *with perfect residue fields k_1 and k_2 of characteristic p respectively. Let $\mathfrak{m}_i$ be the maximal ideal of R_i generated by π_i and ν_i corresponding valuation of R_i for $i = 1, 2$. Let L_i and K_i be the fraction fields of R_i and $W(k_i)$ for $i = 1, 2$ respectively. For any homomorphism $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$, we say that a homomorphism $g : R_1 \longrightarrow R_2$ is a (n_1, n_2)*-lifting *of ϕ at π_1 if g satisfies the following:*

- *There exists a representaive β of $\phi(\pi_1+\mathfrak{m}_1^{n_1})$ which satisfies*
$$\widetilde{\nu_2}\big(g(\pi_1)-\beta\big) > \max_{\sigma}\left\{\widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big)-\beta\Big) : \sigma\big(g(\pi_1)\big)\neq g(\pi_1)\right\}$$
where σ runs through all of $\mathrm{Hom}_{K_2}(L_2, L_2^{alg})$.
- $\phi_{red,1}\circ \mathrm{pr}_{1,1} = \mathrm{pr}_{2,1}\circ g$ *where $\phi_{red,1} : k_1 \longrightarrow k_2$ denotes the natural reduction map of ϕ.*

*When such g is unique, we denote g by $\mathrm{L}_{\pi_1,n_1,n_2}(\phi)$. When $\mathrm{L}_{\pi_1,n_1,n_2}(\phi)$ exists for all $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$, we write $\mathrm{L}_{\pi_1,n_1,n_2} : \mathrm{Hom}(R_{1,n_1}, R_{2,n_2}) \longrightarrow \mathrm{Hom}(R_1, R_2)$. When $n_1 = n_2 = n$, we briefly write $\mathrm{L}_{\pi_1,n_1,n_2} = \mathrm{L}_{\pi_1,n}$ and say that $\mathrm{L}_{\pi_1,n}$ is the n-*lifting at π_1.

If $n_2\nu_1(p)/\nu_2(p) \le n_1$, there is a natural projection map $\mathrm{pr}^{n_1,n_2} : \mathrm{Hom}(R_1, R_2) \longrightarrow \mathrm{Hom}(R_{1,n_1}, R_{2,n_2})$ such that for any g in $\mathrm{Hom}(R_1, R_2)$, $\mathrm{pr}_{2,n_2}\circ g = \mathrm{pr}^{n_1,n_2}(g)\circ \mathrm{pr}_{1,n_1}$. In particular, g is a (n_1, n_2)-lifting of $\mathrm{pr}^{n_1,n_2}(g)$ at π_1. Note that when $n_2 > \nu_2(p)$ and $\mathrm{Hom}(R_{1,n_1}, R_{2,n_2})$ is not empty, $n_2\nu_1(p)/\nu_2(p) \le n_1$ by Lemma 2.5.

The definition of liftings does not depend on the choice of uniformizer. In order to prove this, we need the following lemmas.

Lemma 2.8. *Let R_i be a complete discrete valuation ring of characteristic* 0 *with perfect residue field k_i of characteristic p for $i = 1, 2$. Let $\mathfrak{m}_i$ be the maximal ideal of R_i generated by π_i and S_i the set of Teichmüller representatives of R_i for $i = 1, 2$.*

(1) For any homomorphism $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$, $\phi(S_1+\mathfrak{m}_1^{n_1})$ is contained in $S_2+\mathfrak{m}_2^{n_2}$. Similarly, for any homomorphism $g : R_1 \longrightarrow R_2$, $g(S_1)$ is contained in S_2.

(2) For any homomorphism $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$, $\phi\left((W(k_1)+\mathfrak{m}_1^{n_1})/\mathfrak{m}_1^{n_1}\right)$ is contained in $(W(k_2)+\mathfrak{m}_2^{n_2})/\mathfrak{m}_2^{n_2}$. Similarly, for any homomorphism $g : R_1 \longrightarrow R_2$, $g(W(k_1))$ is contained in $W(k_2)$.

Proof. (1) Since $W(k_i)/pW(k_i) \cong R_i/\mathfrak{m}_i \cong k_i$, S_i is contained in $W(k_i)$ by Lemma 2.4. For any $\lambda \in S_1$, let η_s be any representative of $\phi(\lambda^{1/p^s}+\mathfrak{m}_1^{n_1})$. We note that λ^{1/p^s} exists in S_1 by Lemma 2.4 and $\eta_s^{p^s}+\mathfrak{m}_2^{n_2} = \phi(\lambda+\mathfrak{m}_1^{n_1})$. If θ_s is any other representative of $\phi(\lambda^{1/p^s}+\mathfrak{m}_1^{n_1})$, then $\eta_s-\theta_s \in \mathfrak{m}_2^{n_2}$. Hence, if we write $\eta_s = \theta_s+\pi_2^{n_2}a$ for some a in R_2, the following binomial expansion
$$\begin{aligned}\eta_s^{p^s} &= (\theta_s+\pi_2^{n_2}a)^{p^s}\\ &= \theta_s^{p^s}+p^s\theta_s^{p^s-1}\pi_2^n a+\ldots+(\pi_2^n a)^{p^s}\end{aligned}$$
shows $\eta_s^{p^s}-\theta_s^{p^s} \in \mathfrak{m}_2^s$. Since η_{s+1}^p is a representative of $\phi(\lambda^{1/p^s}+\mathfrak{m}_1^{n_1})$, the calculation above shows that $\{\eta_s^{p^s}\}$ is a Cauchy sequence and $\lim_{s\to\infty}\eta_s^{p^s}$ is well-defined in R_2. Since $\eta_s^{p^s}+\mathfrak{m}_2^{n_2} = \phi(\lambda+\mathfrak{m}_1^{n_1})$ and $\mathfrak{m}_2^{n_2}$ is topologically closed in R_2,
$$\phi\left(\lambda+\mathfrak{m}_1^{n_1}\right) = \left(\lim_{s\to\infty}\eta_s^{p^s}\right)+\mathfrak{m}_2^{n_2}.$$
Similarlly, we have
$$\phi\left(\lambda^{1/p}+\mathfrak{m}_1^{n_1}\right) = \left(\lim_{s\to\infty}\eta_s^{p^{s-1}}\right)+\mathfrak{m}_2^{n_2}.$$
Since
$$\lim_{s\to\infty}\eta_s^{p^s} = \left(\lim_{s\to\infty}\eta_s^{p^{s-1}}\right)^p,$$
we obtain
$$\lim_{s\to\infty}\eta_s^{p^s} \in S_2$$

by Lemma 2.4.

Since $g(\lambda)^{1/p} = g(\lambda^{1/p})$, $g(S_1)$ is contained in S_2 by Lemma 2.4.

(2) For any element a in $W(k_1)$, we can write $a = \sum_{r=0}^{\infty} \lambda_r p^r$ uniquely where λ_r is in S_1 by Lemma 2.4. Then by Lemma 2.8.(1), $g(a) = \sum_{r=0}^{\infty} g(\lambda_r)p^r$ is in $W(k_2)$.

Let $\phi_{res} : (W(k_1) + \mathfrak{m}_1^{n_1})/\mathfrak{m}_1^{n_1} \longrightarrow R_{2,n_2}$ be the restrition map of ϕ to the domain $W(k_1)/(W(k_1) \bigcap \mathfrak{m}_1^{n_1}) \cong (W(k_1) + \mathfrak{m}_1^{n_1})/\mathfrak{m}_1^{n_1}$. By Theorem 2.2, we define g_{res} to be the $(1,1)$-lifting of $\phi_{res,red,1}$ where $\phi_{res,red,1} : k_1 \longrightarrow k_2$ denotes the natural reduction map of ϕ_{res}. We claim that g_{res} induces ϕ_{res}. For any λ in S_1, $g_{res}(\lambda) = \tau$ where τ is a unique representative of $\phi(\lambda + \mathfrak{m}_1^{n_1})$ contained in S_2 by Lemma 2.4. Since g_{res} is a ring homomorphism, $g_{res}(a) = \sum_{r=0}^{\infty} \tau_r p^r$ where τ_r is a unique representative of $\phi_{res}(\lambda_r + \mathfrak{m}_1^{n_1})$ which is contained in S_2. This shows

$$\begin{aligned} g_{res}(a) + \mathfrak{m}^{n_2} &= \left(\sum_{r=0}^{\infty} \tau_r p^r\right) + \mathfrak{m}_2^{n_2} \\ &= \sum_{r=0}^{\infty} p^r \phi_{res}(\lambda_r + \mathfrak{m}_1^{n_1}) \\ &= \phi_{res}(a + \mathfrak{m}_1^{n_1}), \end{aligned}$$

and hence, g_{res} induces ϕ_{res}. Since the image of g_{res} is contained in $W(k_2)$, the image of ϕ_{res} is contained in $(W(k_2) + \mathfrak{m}_2^{n_2})/\mathfrak{m}_2^{n_2}$.

□

Lemma 2.9. *Let R_1 and R_2 be complete discrete valuation rings of characteristic 0 with perfect residue fields k_1 and k_2 of characteristic p respectively. Let $\mathfrak{m}_i$ be the maximal ideal of R_i generated by π_i and ν_i corresponding valuation of R_i for $i = 1,2$. Let L_i and K_i be the fraction fields of R_i and $W(k_i)$ for $i = 1,2$ respectively.*

(1) Let α be a uniformizer of R_1 other than π_1. Then $M(R_1)_{\pi_1} = M(R_1)_\alpha$. We briefly write $M(R_1)_{\pi_1} = M(R_1)$.

(2) Suppose $[L_1 : K_1] = [L_2 : K_2] = e$, that is, $\nu_1(p) = \nu_2(p) = e$. Suppose there is an isometry $g : L_1 \longrightarrow L_2$. Then $M(R_1) = M(R_2)$.

Proof. (1) By Lemma 2.4, we can write $\alpha = \sum_{r=1}^{\infty} \lambda_r \pi_1^r$ where λ_r is a Teichmüller representative of R_1 for each r and $\lambda_1 \neq 0$. Since $R_1/\mathfrak{m}_1 = k_1$, λ_r is in $W(k_1)$ for each r by Lemma 2.4. For any σ in $\mathrm{Hom}_{K_1}(L_1, K_1^{alg})$,

$$\begin{aligned} \alpha - \sigma(\alpha) &= \sum_{r=1}^{\infty} \lambda_r \pi_1^r - \sigma\left(\sum_{r=1}^{\infty} \lambda_r \pi_1^r\right) \\ &= \sum_{r=1}^{\infty} \lambda_r \Big(\pi_1^r - \sigma(\pi_1^r)\Big) \\ &= \Big(\pi_1 - \sigma(\pi_1)\Big) \sum_{r=1}^{\infty} \lambda_r \left(\sum_{j=0}^{r-1} \pi_1^{r-1-j} \sigma(\pi_1^j)\right) \end{aligned}$$

shows $\widetilde{\nu_1}(\alpha - \sigma(\alpha)) = \widetilde{\nu_1}(\pi_1 - \sigma(\pi_1))$ since

$$\widetilde{\nu_1}\left(\sum_{r=1}^{\infty} \lambda_r \left(\sum_{j=0}^{r-1} \pi_1^{r-1-j} \sigma(\pi_1^j)\right)\right) = 0.$$

We have $M(R_1)_{\pi_1} = M(R_1)_\alpha$.

(2) By Lemma 2.8.(2), $g(K_1)$ is contained in K_2. Let f_1 be the monic irreducible polynomial of π_1 over $W(k_1)$. Since g is an isometry, we have $\overline{\nu_2}(g(\pi_1)) = \overline{\nu_1}(\pi_1) = 1/e$, and hence, $g(\pi_1)$ is a uniformizer of L_2. Let $\widetilde{g} : L_1^{alg} \longrightarrow L_2^{alg}$ be an extended homomorphism of g. Let $g(f_1)$ be the monic irreducible polynomial of $g(\pi_1)$ over K_2. If we write $f_1 = x^e + ...a_1x + a_0$, we have

$$g(f_1) = x^e + ...g(a_1)x + g(a_0)$$

since $g(K_1)$ is contained in K_2. Then by Lemma 2.9.(1) and Lemma 2.6,

$$\begin{aligned}
M(R_2) &= \max\left\{\widetilde{\nu_2}\big(g(\pi_1) - \eta\big) : g(f_1)(\eta) = 0, \eta \neq g(\pi_1)\right\} \\
&= \max\left\{\widetilde{\nu_2}\big(g(\pi_1) - \widetilde{g}(\pi_1')\big) : f_1(\pi_1') = 0, \pi_1' \neq \pi_1\right\} \\
&= \max\left\{\widetilde{\nu_1}(\pi_1 - \pi_1') : f_1(\pi_1') = 0, \pi_1' \neq \pi_1\right\} \\
&= M(R_1)
\end{aligned}$$

□

Proposition 2.10. *Let R_1 and R_2 be complete discrete valuation rings of characteristic 0 with perfect residue fields of characteristic p. Let $\mathfrak{m}_i$ be the maximal ideal of R_i generated by π_i and ν_i corresponding valuation of R_i for $i = 1, 2$. Let L_i and K_i be the fraction fields of R_i and $W(k_i)$ for $i = 1, 2$ respectively.*

(1) Let $g : R_1 \longrightarrow R_2$ be a (n_1, n_2)-lifting of $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$ at π_1 which satisfies

$$\widetilde{\nu_2}\big(g(\pi_1) - \beta\big) > \max_\sigma \left\{\widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big) - \beta\Big) : \sigma\big(g(\pi_1)\big) \neq g(\pi_1)\right\}$$

where σ runs through all of $\mathrm{Hom}_{K_2}(L_2, L_2^{alg})$ and β is a representative of $\phi(\pi_1 + \mathfrak{m}_1^{n_1})$. Then

$$\max_\sigma \left\{\widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big) - \beta\Big) : \sigma\big(g(\pi_1)\big) \neq g(\pi_1)\right\} = M(R_1).$$

(2) The definition of liftings is independent of the choice of uniformizer of R_1. More precisely, saying that $g : R_1 \longrightarrow R_2$ is a (n_1, n_2)-lifting of $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$ at π_1 is equivalent to the following:

- *For any x in R_1, there exists a representative β_x of $\phi(x + \mathfrak{m}_1^{n_1})$ which satisfies*

$$\widetilde{\nu_2}\big(g(x) - \beta_x\big) > M(R_1)$$

- $\phi_{red,1} \circ \mathrm{pr}_{1,1} = \mathrm{pr}_{2,1} \circ g$

We write $\mathrm{L}_{\pi_1,n_1,n_2} = \mathrm{L}_{n_1,n_2}$ and say that L_{n_1,n_2} is the (n_1, n_2)-lifting.

Proof. (1) For $\sigma \in \mathrm{Hom}_{K_2}(L_2, L_2^{alg})$ with $\sigma(g(\pi_1)) \neq g(\pi_1)$,

$$\begin{aligned}
\widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big) - \beta\Big) &= \widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big) - g(\pi_1) + g(\pi_1) - \beta\Big) \\
&= \min\left\{\widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big) - g(\pi_1)\Big), \widetilde{\nu_2}\big(g(\pi_1) - \beta\big)\right\} \\
&= \widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big) - g(\pi_1)\Big)
\end{aligned}$$

Since $\widetilde{\nu_2}(g(\pi_1)-\beta) > \widetilde{\nu_2}(\sigma(g(\pi_1))-\beta)$. This shows

$$\begin{aligned} M(R_1) &= \max_\sigma \left\{ \widetilde{\nu_1}\big(\pi_1 - \sigma(\pi_1)\big) : \sigma(\pi_1) \neq \pi_1 \right\} \\ &= \max_\sigma \left\{ \widetilde{\nu_2}\Big(g(\pi_1) - \sigma\big(g(\pi_1)\big)\Big) : \sigma\big(g(\pi_1)\big) \neq g(\pi_1) \right\} \\ &= \max_\sigma \left\{ \widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big) - \beta\Big) : \sigma\big(g(\pi_1)\big) \neq g(\pi_1) \right\} \end{aligned}$$

where the second equality follows from Lemma 2.9.(2) since $[K_2(g(\pi_1)) : K_2]$ is equal to $[L_1 : K_1]$ and $g(\pi_1)$ is a uniformizer of $K_2(g(\pi_1))$.

(2) Let $g : R_1 \longrightarrow R_2$ be a (n_1, n_2)-lifting of $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$ at π_1 which satisfies

$$\widetilde{\nu_2}\big(g(\pi_1) - \beta\big) > \max_\sigma \left\{ \widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big) - \beta\Big) : \sigma\big(g(\pi_1)\big) \neq g(\pi_1) \right\}$$

where σ runs through all of $\mathrm{Hom}_{K_2}(L_2, L_2^{alg})$ and β is a representative of $\phi(\pi_1 + \mathfrak{m}_1^{n_1})$. For any x in R_1, we can write $x = \sum_{r=0}^{\infty} \lambda_r \pi_1^r$ where λ_r is in the set S_1 of Teichmüller representatives for each r. Then

$$\phi(x + \mathfrak{m}_1^{n_1}) = \phi\left(\left(\sum_{r=0}^{\infty} \lambda_r \pi_1^r\right) + \mathfrak{m}_1^{n_1}\right) = \left(\sum_{r=0}^{\infty} \tau_r \beta^r\right) + \mathfrak{m}_2^{n_2}$$

where τ_r is a representative of $\phi(\lambda_r + \mathfrak{m}_1^{n_1})$ contained in S_2 guaranteed by Lemma 2.8.(1). In particular $\sum_{r=0}^{\infty} \tau_r \beta^r$ is a representative of $\phi(x + \mathfrak{m}_1^{n_1})$, say β_x. By the second condition of the definition of liftings and Lemma 2.4, we have $g(\lambda_r) = \tau_r$, and hence,

$$g(x) = g\left(\sum_{r=0}^{\infty} \lambda_r \pi_1^r\right) = \sum_{r=0}^{\infty} \tau_r g(\pi_1)^r.$$

We obtain

$$\begin{aligned} \widetilde{\nu_2}(g(x) - \beta_x) &= \widetilde{\nu_2}\left(\sum_{r=0}^{\infty} \tau_r g(\pi_1)^r - \sum_{r=0}^{\infty} \tau_r \beta^r\right) \\ &= \widetilde{\nu_2}\left(\Big(g(\pi_1) - \beta\Big) \sum_{r=1}^{\infty} \tau_r \left(\sum_{j=0}^{r-1} g(\pi_1)^{r-1-j} \beta^j\right)\right) \\ &> M(R_1) \end{aligned}$$

since

$$\begin{aligned} \widetilde{\nu_2}\big(g(\pi_1) - \beta\big) &> \max_\sigma \left\{ \widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big) - \beta\Big) : \sigma\big(g(\pi_1)\big) \neq g(\pi_1) \right\} \\ &= M(R_1). \end{aligned}$$

□

Lemma 2.11 (Krasner's lemma). *Let (K, ν) be henseilan valued field whose value group is contained in $\mathbb{R}$ and let $a, b \in K^{alg}$. Suppose a is separable over $K(b)$. Suppose that for all embeddings $\sigma (\neq id)$ of $K(a)$ over K, we have*

$$\widetilde{\nu}(b - a) > \widetilde{\nu}\big(\sigma(a) - a\big).$$

Then $K(a) \subset K(b)$.

Proof. Chapter 2 of [24]. □

The following theorem shows that there is a unique lifting if we enlarge the lengths of residue rings.

Theorem 2.12. *Let R_1 and R_2 be complete discrete valuation rings of characteristic 0 with perfect residue fields k_1 and k_2 of characteristic p respectively. Let $\mathfrak{m}_i$ be the maximal ideal of R_i generated by π_i and ν_i corresponding valuation of R_i for $i = 1, 2$. Let L_i and K_i be the fraction fields of R_i and $W(k_i)$ for $i = 1, 2$ respectively. Suppose $n_2 > M(R_1)\nu_1(p)\nu_2(p)$ and $\mathrm{Hom}(R_{1,n_1}, R_{2,n_2})$ is not empty. Then there exists a unique (n_1, n_2)-lifting $\mathrm{L}_{n_1,n_2} : \mathrm{Hom}(R_{1,n_1}, R_{2,n_2}) \longrightarrow \mathrm{Hom}(R_1, R_2)$. $\mathrm{L}_{n_1,n_2}(\phi)$ is also an isomorphism when ϕ is an isomorphism.*

Proof. Let S_i be the set of Teichmüller representatives of R_i. By Lemma 2.8.(2), let $\phi_{res} : (W(k_1) + \mathfrak{m}_1^{n_1})/\mathfrak{m}_1^{n_1} \longrightarrow (W(k_2) + \mathfrak{m}_2^{n_2})/\mathfrak{m}_2^{n_2}$ be the restrition map of ϕ. For an element $a = \sum_{r=0}^{\infty} \lambda_r p^r$ in $W(k_1)$, as in the proof of Lemma 2.8.(2), we define $g_{res} : W(k_1) \longrightarrow W(k_2)$ by $g_{res}(a) = \sum_{r=0}^{\infty} \tau_r p^r$ where τ_r is a unique representative of $\phi_{res}(\lambda_r + \mathfrak{m}_1^{n_1})$ which is contained in S_2. Then g_{res} induces ϕ_{res}. By Lemma 2.3, $L_1 = K_1(\alpha)$ is totally ramified of degree $\nu_1(p)$ over K_1 where $\alpha = \pi_1$. Let f be the monic irreducible polynomial of α over K_1. The ring homomorphism g_{res} induces the field homomorphism from K_1 into K_2. We still denote the fraction field homomorphism by g_{res} if there is no confusion. Then $g_{res} : K_1 \longrightarrow K_2$ is an isometry. Let $\widetilde{g_{res}} : K_1^{alg} \longrightarrow K_2^{alg}$ be an extended field homomorphism of g_{res}. Then $\widetilde{g_{res}}$ is an isometry by Lemma 2.6. Let $g_{res}(f)$ be the monic irreducible polynomial of $\widetilde{g_{res}}(\alpha)$ over K_2. If we write

$$\begin{aligned} f =& x^{\nu_1(p)} + ... + a_1 x + a_0 \\ =& (x - \alpha_1)...(x - \alpha_{\nu_1(p)}), \end{aligned}$$

where $\alpha = \alpha_1$, then

$$\begin{aligned} g_{res}(f) =& x^{\nu_1(p)} + ... + g_{res}(a_1)x + g_{res}(a_0) \\ =& \big(x - \widetilde{g_{res}}(\alpha_1)\big)...\big(x - \widetilde{g_{res}}(\alpha_{\nu_1(p)})\big) \end{aligned}$$

since $[K_2(\widetilde{g_{res}}(\alpha)) : K_2] \le \nu_1(p)$ and $\widetilde{\nu_2}(\widetilde{g_{res}}(\alpha)) = 1/\nu_1(p)$. Let β be any representative of $\phi(\alpha + \mathfrak{m}_1^{n_1})$. Since g_{res} induces ϕ_{res}, we can write

$$\begin{aligned} 0 + \mathfrak{m}_2^{n_2} &= \phi(f(\alpha) + \mathfrak{m}_1^{n_1}) \\ &= \phi(\alpha + \mathfrak{m}_1^{n_1})^{\nu_1(p)} + ... + \phi(a_1 + \mathfrak{m}_1^{n_1})\phi(\alpha + \mathfrak{m}_1^{n_1}) + \phi(a_0 + \mathfrak{m}_1^{n_1}) \\ &= g_{res}(f)(\beta) + \mathfrak{m}_2^{n_2}. \end{aligned}$$

This shows that $g_{res}(f)(\beta)$ is in $\mathfrak{m}_2^{n_2}$ and

$$\nu_2\big(g_{res}(f)(\beta)\big) \ge n_2 > M(R_1)\nu_1(p)\nu_2(p).$$

We claim that there exists an index i_0 satisfying $\widetilde{\nu_2}(\beta - \widetilde{g_{res}}(\alpha_{i_0})) > M(R_1)$. If $\widetilde{\nu_2}(\beta - \widetilde{g_{res}}(\alpha_i)) \le M(R_1)$ for all i, then

$$\widetilde{\nu_2}\big(g_{res}(f)(\beta)\big) = \widetilde{\nu_2}\left(\prod_i \Big(\beta - \widetilde{g_{res}}(\alpha_i)\Big)\right) \le M(R_1)\nu_1(p).$$

This shows

$$\nu_2\big(g_{res}(f)(\beta)\big) = \nu_2(p)\widetilde{\nu_2}\big(g_{res}(f)(\beta)\big) \le M(R_1)\nu_1(p)\nu_2(p).$$

Thus there is an index i_0 satisfying

$$\widetilde{\nu_2}\big(\beta - \widetilde{g_{res}}(\alpha_{i_0})\big) > M(R_1) = \max\left\{\widetilde{\nu_2}\big(\widetilde{g_{res}}(\alpha_1) - \widetilde{g_{res}}(\alpha_j)\big) : j = 2, ..., \nu_1(p)\right\}$$

where the equality follows from the fact that $\widetilde{g_{res}}$ is an isometry. Hence, Krasner's lemma 2.11 shows $K_2(\widetilde{g_{res}}(\alpha_{i_0})) \subset K_2(\beta) \subset L_2$. We define an extended homomorphism $g : L_1 \longrightarrow L_2$ of $g_{res} : K_1 \longrightarrow K_2$ by the rule $\pi_1 \mapsto g(\pi_1) = \widetilde{g_{res}}(\alpha_{i_0})$. g induces the restricted homomorphism from R_1 to R_2 which is still denoted by g. Since g_{res} induces ϕ_{res} and

$$M(R_1) = \max_{\sigma} \left\{ \widetilde{\nu_2}\Big(\sigma\big(g(\pi_1)\big) - \beta\Big) : \sigma\big(g(\pi_1)\big) \neq g(\pi_1) \right\}$$

by Lemma 2.10.(1), g is a (n_1, n_2)-lifting of ϕ.

Suppose that $g_1 : R_1 \longrightarrow R_2$ is a (n_1, n_2)-lifting of ϕ other than g. Then we have

$$\widetilde{\nu_2}\big(g_1(\pi_1) - \beta\big) > \max_{\sigma} \left\{ \widetilde{\nu_2}\Big(\sigma\big(g_1(\pi_1)\big) - \beta\Big) : \sigma\big(g\,(\pi_1)\,\big) \neq g(\pi_1) \right\}$$

by the first condition of the definition of liftings. By the second condition of the definition of liftings and by Theorem 2.2, we obtain the restriction $g_1|_{W(k_1)}$ of g_1 to $W(k_1)$ is equal to $g|_{W(k_1)}$. This shows that the monic irreducible polynomial $g_1(f)$ of $g_1(\pi_1)$ is equal to the monic irreducible polynomial $g(f)$ of $g(\pi_1)$ and

$$\left\{ \sigma\big(g_1(\pi_1)\big) : \sigma \in \mathrm{Hom}_{K_2}(L_2, L_2^{alg}) \right\} = \left\{ \sigma\big(g(\pi_1)\big) : \sigma \in \mathrm{Hom}_{K_2}(L_2, L_2^{alg}) \right\}.$$

In particular $g_1(\pi_1) = \sigma(g(\pi_1))$ for some $\sigma \in \mathrm{Hom}_{K_2}(L_2, L_2^{alg})$. But since $g_1(\pi_1) \neq g(\pi_1)$, we have the inequalities $\widetilde{\nu_2}(g_1(\pi_1) - \beta) > \widetilde{\nu_2}(g(\pi_1) - \beta)$ and $\widetilde{\nu_2}(g_1(\pi_1) - \beta) < \widetilde{\nu_2}(g(\pi_1) - \beta)$ simultaneously. This gives a contradiction. Hence we obtain the uniqueness of the lifting.

When ϕ is an isomorphism, so are ϕ_{res} and g_{res}. We obtain $[L_2 : K_2] = [L_1 : K_1]$, and hence, $\mathrm{L}_{n_1,n_2}(\phi)$ is also an isomorphism.

□

We note that the proof of Theorem 2.12 works for any representative β of $\phi(\pi_1 + \mathfrak{m}_1^{n_1})$.

Example 2.13. *(1) Let $R_1 = \mathbb{Z}_3[\sqrt{3}]$ and $R_2 = \mathbb{Z}_3[\sqrt{-3}]$. There is no homomorphism between R_1 and R_2 by Kummer theory. But there is an isomorphism*

$$\phi : R_{1,2} = \frac{\mathbb{Z}_3[\sqrt{3}]}{3\mathbb{Z}_3[\sqrt{3}]} \longrightarrow R_{2,2} = \frac{\mathbb{Z}_3[\sqrt{-3}]}{3\mathbb{Z}_3[\sqrt{-3}]}$$

given by the rule $a + b\sqrt{3} \mapsto a + b\sqrt{-3}$. Since $\nu_1(3) = \nu_2(3) = 2$ and $M(R_1) = \widetilde{\nu_1}(\sqrt{3} - (-\sqrt{3})) = 1/2$, we obtain $M(R_1)\nu_1(3)\nu_2(3) = 2$. Hence the lower bound for n_2 in Theorem 2.12 *is the best possible in this case. This phenomenon will be generalized in* Proposition 3.11.

(2) If we take $R_1 = R_2 = \mathbb{Z}_3[\sqrt{3}]$ and $n_1 = n_2 = 2n$, then $R_{1,2n} = R_{2,2n} \cong (\mathbb{Z}_3/3^n\mathbb{Z}_3)[x]/(x^2 - 3)$. Then $\phi : a + bx \mapsto a + (1 + 3^{n-1})bx = \phi(a + bx)$ defines an isomorphism between R_1 and R_2. But when $n > 1$, there is no homomorphism $g : R_1 \longrightarrow R_2$ which induces ϕ since Galois conjugates of $\sqrt{3}$ are $\pm\sqrt{3}$. This shows that in Theorem 2.12, *we can not guarantee that the following diagram is commutative:*

$$\begin{array}{ccc} R_1 & \xrightarrow{\mathrm{L}_{n_1,n_2}(\phi)} & R_2 \\ \downarrow & & \downarrow \\ R_{1,n_1} & \xrightarrow{\phi} & R_{2,n_2} \end{array}$$

Remark 2.14. *(1) We regard* Theorem 2.12 *as a generalization of* Theorem 2.2.(2)*. We can restate* Theorem 2.2.(2) *as follows. For* $\phi : k_1 \longrightarrow k_2$*, there exists a unique homomorphism* $g : W(k_1) \longrightarrow W(k_2)$ *which is characterized by the following property:*

- *For any* x *in* $W(k_1)$*, there exists a representative* β_x *of* $\phi(x+pW(k_1))$ *which satisfies* $\nu_2(g(x) - \beta_x) = \infty$*.*

In general, the property above does not hold as is seen in Example 2.13.(2)*. But we can restate* Theorem 2.12 *as follows. For* $\phi : R_{1,n_1} \longrightarrow R_{2,n_2}$*, there exists a unique homomorphism* $g : R_1 \longrightarrow R_2$ *which is characterized by the following property:*

- *There exists* N *depending on* R_1 *only such that for any* x *in* R_1*, there exists a representative* β_x *of* $\phi(x+\mathfrak{m}_1^{n_1})$ *which satisfies* $\nu_2(g(x)-\beta_x) > N$*.*

This follows from Proposition 2.10.(2)*.*

(2) Suppose $R_1 = R_2 = R$*,* $k_1 = k_2 = k$*,* $n_1 = n_2 = n$*, and* $\nu_1 = \nu_2 = \nu$*. When* k *is finite and* ϕ *is an isomorphism, Basarab(*[6]*) claimed that if* $n > M(R)\nu(p)$*,* Theorem 2.12 *should hold by Krasner's lemma. But it is not correct by* Example 2.13.(1)*. Moreover, there is a gap in the argument with respect to the choice of uniformizer in* [6]*.*

In Theorem 2.12, $M(R)e(R)$ can vary when R changes. The following lemma will play an important role for bounding $M(R)e(R)$.

Lemma 2.15. *Let* $R \subset S$ *be discrete valuation rings and* S *a finitely generated* R*-module. Suppose* $S = R[\alpha]$ *for some* α *in* S*. Let* $f(x)$ *in* $R[x]$ *be the monic irreducible polynomial of* α *over* R*.*

(1) The different $\mathfrak{D}_{S/R}$ *of* S/R *is a principal ideal generated by* $f'(\alpha)$

(2) Let $\mathfrak{B}$ *be the maximal ideal of* S*. Let* e *be the ramification index of* S *over* R *and* ν_S *the valuation corresponding to* S*. Let* s *be the power which satisfies* $\mathfrak{B}^s = \mathfrak{D}_{S/R}$*. Then one has*

$$\begin{aligned} s &= e - 1 && \text{if } S \text{ is tamely ramified,} \\ e \le s &\le e - 1 + \nu_S(e) && \text{if } S \text{ is wildly ramified.} \end{aligned}$$

Proof. Chapter 3, Section 2 of [25]. □

The following theorem can be regarded as a generalized version of Theorem 2.2.(1) for the ramified case.

Theorem 2.16. *Let* $\overline{R}$ *be a principal Artinian local ring of length* n *with perfect residue field* k *of characteristic* p *and maximal ideal* $\overline{\mathfrak{m}}$*. Here length* n *means* $\overline{\mathfrak{m}}^n = 0$ *and* $\overline{\mathfrak{m}}^{n-1} \neq 0$ *which is denoted by* $l(\overline{R}) = n$*. Suppose that* $\overline{R}$ *has no finite subfield as a subring. For any positive integer* a*, if* a *generates an ideal* $\overline{\mathfrak{m}}^k$*, we denote* k *by* $\nu(a)$*. Suppose*

$$l(\overline{R}) = n > \nu(p) + \nu(p)\nu\big(\nu(p)\big).$$

Then there exists a complete discrete valuation ring of characteristic 0 *which has* $\overline{R}$ *as its* n*-th residue ring. Such a ring is unique up to isomorphism.*

Proof. Any principal Artinian local ring is a homomorphic image of a discrete valuation ring. This can be proved by Cohen structure theorem for complete local rings([19]) or, more directly, by the property of CPU-rings([17]). Since the completion of a discrete valuation ring R has the same n-th residue ring as that of R, we may assume that there are complete discrete valuation rings R_1 and R_2 which have $\overline{R}$ as isomorphic copies of $R_{1,n}$ and $R_{2,n}$ respectively. We note that R_i is of characteristic 0 for $i = 1, 2$ since $\overline{R}$ has no finite subfield as a subring. Let L_i and K_i be the fraction fields of R_i and $W(k_i)$ for $i = 1, 2$ respectively. Then by Lemma 2.3, $L_1 = K_1(\alpha)$ where $\alpha = \pi_1$ is a uniformizer of R_1. Let f be the monic irreducible polynomial of α over K_1. Then one can write

$$\begin{aligned} f =& x^{\nu(p)} + ... + a_1 x + a_0 \\ =& (x-\alpha_1)...(x-\alpha_{\nu(p)}) \end{aligned}$$

where $\alpha = \alpha_1$. Let ν_i be the corresponding valuation of R_i. We note that $\nu_1(p) = \nu_2(p) = \nu(p)$ since $\overline{R}$ has no finite subfield as a subring. We consider the differentiation f' of f. There are two cases.

- Tame case: Suppose L_1/K_1 is tamely ramified. Hence, $\nu(\nu(p)) = 0$. For all distinct i and j, $\widetilde{\nu_1}(\alpha_i) = 1/\nu(p)$ and hence $\widetilde{\nu_1}(\alpha_i - \alpha_j) \geq 1/\nu(p)$. We obtain

$$\begin{aligned} \widetilde{\nu_1}\big(f'(\alpha_1)\big) &= \widetilde{\nu_1}\left(\prod_{j\neq 1}(\alpha_1 - \alpha_j)\right) \\ &= \sum_{j \neq 1} \widetilde{\nu_1}(\alpha_1 - \alpha_j) \\ &\geq \frac{\nu(p)-1}{\nu(p)}. \end{aligned}$$

  Since

$$\widetilde{\nu_1}\big(f'(\alpha_1)\big) = \frac{\nu(p)-1}{\nu(p)}$$

  by Lemma 2.15, $\widetilde{\nu_1}(\alpha_1 - \alpha_j) = 1/\nu(p) = M(R_1)$ for $j \neq 1$. Hence we have

$$\begin{aligned} \nu(p) + \nu(p)\nu\big(\nu(p)\big) &= \nu(p) \\ &= M(R_1)\nu(p)^2 \end{aligned}$$

  and Theorem 2.12 finishes the proof.
- Wild case: Suppose L_1/K_1 is wildly ramified. Since $\widetilde{\nu_1}(\alpha_i - \alpha_j) \geq 1/\nu(p)$ for all distinct i and j,

$$\begin{aligned} M(R_1) &\leq \widetilde{\nu_1}\big(f'(\alpha_1)\big) - \frac{\nu(p)-2}{\nu(p)} \\ &\leq \frac{\nu(p) - 1 + \nu\big(\nu(p)\big)}{\nu(p)} - \frac{\nu(p)-2}{\nu(p)} \\ &= \frac{1 + \nu\big(\nu(p)\big)}{\nu(p)} \end{aligned}$$

  by Lemma 2.15, and hence, $M(R_1)\nu(p)^2 \leq \nu(p) + \nu(p)\nu(\nu(p))$. Again Theorem 2.12 finishes the proof.

□

Note that the notation $\nu(p)$ in Theorem 2.16 is compatible with the previously defined valuation. Suppose that a discrete valuation ring R with valuation ν and maximal ideal $\mathfrak{m}$ has $\overline{R}$ as its residue ring. Then $\nu(p)$ is equal to the power of the maximal ideal generated by p, that is, $Rp = \mathfrak{m}^{\nu(p)}$ as we noted in the proof of Theorem 2.16.

3. Functoriality

For a prime number p, let $\mathcal{C}_p$ be a category consisting of the following data :

- $\mathrm{Ob}(\mathcal{C}_p)$ is the family of absolutely unramified complete discrete valuation rings of mixed characteristic having perfect residue fields of characteristic p.
- $\mathrm{Mor}_{\mathcal{C}_p}(R_1, R_2) := \mathrm{Hom}(R_1, R_2)$ for R_1 and R_2 in $\mathrm{Ob}(\mathcal{C}_p)$.

Let $\mathcal{R}_p$ be a category consisting of the following data :

- $\mathrm{Ob}(\mathcal{R}_p)$ is the family of perfect fields of characteristic p.
- $\mathrm{Mor}_{\mathcal{R}_p}(k_1, k_2) := \mathrm{Hom}(k_1, k_2)$ for k_1 and k_2 in $\mathrm{Ob}(\mathcal{R}_p)$.

Let $\mathrm{Pr} : \ \mathcal{C}_p \to \mathcal{R}_p$ be the canonical projection functor. We restate Theorem 2.2 categorically as follows :

Theorem 3.1. *There exists a functor* $\mathrm{L} : \mathcal{R}_p \longrightarrow \mathcal{C}_p$ *which satisfies:*

- *The composite functor* $\mathrm{Pr} \circ \mathrm{L}$ *is equivalent to the identity functor* $\mathrm{Id}_{\mathcal{R}_p}$.
- *The composite functor* $\mathrm{L} \circ \mathrm{Pr}$ *is equivalent to the identity functor* $\mathrm{Id}_{\mathcal{C}_p}$.

The main purpose of this section is to give a generalized version of Theorem 3.1 for the ramified case. For a prime number p and a positive integer e, let $\mathcal{C}_{p,e}$ be a category consisting of the following data :

- $\mathrm{Ob}(\mathcal{C}_{p,e})$ is the family of complete discrete valuation rings of mixed characteristic having perfect residue fields of characteristic p and the ramification index e; and
- $\mathrm{Mor}_{\mathcal{C}_{p,e}}(R_1, R_2) := \mathrm{Hom}(R_1, R_2)$ for R_1 and R_2 in $\mathrm{Ob}(\mathcal{C}_{p,e})$.

Let $\mathcal{R}^n_{p,e}$ be a category consisting of the following data :

- For $n \leq e$, $\mathrm{Ob}(\mathcal{R}^n_{p,e})$ is the family of principal Artinian local rings $\overline{R}$ of length n with perfect residue fields of characteristic p, and for $n > e$, $\mathrm{Ob}(\mathcal{R}^n_{p,e})$ is the family of principal Artinian local rings $\overline{R}$ of length n with perfect residue fields of characteristic p such that $p \in \overline{\mathfrak{m}}^e \setminus \overline{\mathfrak{m}}^{e+1}$ where $\overline{\mathfrak{m}}$ is the maximal ideal of $\overline{R}$; and
- $\mathrm{Mor}_{\mathcal{R}^n_{p,e}}(\overline{R_1}, \overline{R_2}) := \mathrm{Hom}(\overline{R_1}, \overline{R_2})$ for $\overline{R_1}$ and $\overline{R_2}$ in $\mathrm{Ob}(\mathcal{R}^n_{p,e,})$,

Note that for $e_1, e_2 \geq 1$ and for $n \leq e_1, e_2$, two categories $\mathcal{R}^n_{p,e_1}, \mathcal{R}^n_{p,e_2}$ are the same. For each $m > n$, let $\mathrm{Pr}_n : \ \mathcal{C}_{p,e} \to \mathcal{R}^n_{p,e}$ and $\mathrm{Pr}^m_n : \ \mathcal{R}^m_{p,e} \to \mathcal{R}^n_{p,e}$ be the canonical projection functors respectively.

Definition 3.2. *Fix a prime number* p *and a positive integer* e.

1. *We say that the category* $\mathcal{C}_{p,e}$ *is* n-liftable *if there is a functor* $\mathrm{L} : \mathcal{R}^n_{p,e} \longrightarrow \mathcal{C}_{p,e}$ *which satisfies the following:*
   - $(\mathrm{Pr}_n \circ \mathrm{L})(\overline{R}) \cong \overline{R}$ *for each* $\overline{R}$ *in* $\mathrm{Ob}(\mathcal{R}_{p,e})$.
   - $\mathrm{Pr}_1 \circ \mathrm{L}$ *is equivalent to* Pr^n_1.
   - $\mathrm{L} \circ \mathrm{Pr}_n$ *is equivalent to* $\mathrm{Id}_{\mathcal{C}_{p,e}}$.

   We say that L *is a* n-th lifting functor *of* $\mathcal{C}_{p,e}$.
2. *The* lifting number *for* $\mathcal{C}_{p,e}$ *is the smallest positive integer* n *such that* $\mathcal{C}_{p,e}$ *is* n*-liftable. If there is no such* n*, we define the lifting number for* $\mathcal{C}_{p,e}$ *to be* ∞.

Remark 3.3. *(1) In* Definition 3.2*, the restriction of* L *to* $\mathrm{Iso}(R_n)$ *is a surjective group homomorphism from* $\mathrm{Iso}(R_n)$ *to* $\mathrm{Iso}(R)$ *for each* $R \in \mathrm{Ob}(\mathcal{C}_{p,e})$*.*

(2) Suppose that there is a n-th lifting functor $\mathrm{L} : \mathcal{R}^n_{p,e} \to \mathcal{C}_{p,e}$*. For any* $\overline{R}$ *in* $\mathrm{Ob}(\mathcal{R}_{p,e})$*, up to isomorphism,* $\mathrm{L}(\overline{R})$ *is a unique object in* $\mathrm{Ob}(\mathcal{C}_{p,e})$ *which has* $\overline{R}$ *as its n-th residue ring. Suppose that* R *in* $\mathrm{Ob}(\mathcal{C}_{p,e})$ *has* $\overline{R}$ *as its n-th residue ring. Since* $\mathrm{L} \circ \mathrm{Pr}_n$ *is equivalent to the identity functor* $\mathrm{Id}_{\mathcal{C}_{p,e}}$*,* $R = \mathrm{Id}_{\mathcal{C}_{p,e}}(R)$ *is isomorphic to* $(\mathrm{L} \circ \mathrm{Pr}_n)(R) = \mathrm{L}(\overline{R})$*.*

(3) The lifting number for $\mathcal{C}_p$ *is* 1 *by* Theorem 3.1*. We will see that the lifting number for* $\mathcal{C}_{p,e}$ *is always larger than e whenever* $e > 1$ *in* Corollary 3.17*. For* $n \geq e$*, we have that a functor* $\mathrm{L}_{n+1} := \mathrm{L}_n \circ \mathrm{Pr}^{n+1}_n$ *is a* $(n+1)$*-th lifting functor of* $\mathcal{C}_{p,e}$ *for any n-th lifting functor* $\mathrm{L}_n : \mathcal{R}^n_{p,e} \to \mathcal{C}_{p,e}$*. For* $\overline{R}$ *in* $\mathrm{Ob}(\mathcal{R}^{n+1}_{p,e})$*, there exists a ring* R *in* $\mathrm{Ob}(\mathcal{C}_{p,e})$ *which satisfies* $\mathrm{Pr}_{n+1}(R) = \overline{R}$ *as noted in the proof of* Theorem 2.16*. Since there is a unique object in* $\mathrm{Ob}(\mathcal{C}_{p,e})$ *which has* $\mathrm{Pr}_n(R)$ *as its n-th residue ring by* Remark 3.3.(2)*, we have*

$$(\mathrm{Pr}_{n+1} \circ \mathrm{L}_{n+1})\left(\overline{R}\right) = \mathrm{Pr}_{n+1} \circ (\mathrm{L}_n \circ \mathrm{Pr}^{n+1}_n)\left(\overline{R}\right) = \mathrm{Pr}_{n+1}(R) = \overline{R}.$$

$\mathrm{Pr}_1 \circ \mathrm{L}_{n+1} = (\mathrm{Pr}_1 \circ \mathrm{L}_n) \circ \mathrm{Pr}^{n+1}_n$ *is equivalent to* $\mathrm{Pr}^n_1 \circ \mathrm{Pr}^{n+1}_n = \mathrm{Pr}^{n+1}_1$ *and* $\mathrm{L}_{n+1} \circ \mathrm{Pr}_{n+1} = (\mathrm{L}_n \circ \mathrm{Pr}^{n+1}_n) \circ \mathrm{Pr}_{n+1} = \mathrm{L}_n \circ \mathrm{Pr}_n$ *is equivalent to* $\mathrm{Id}_{\mathcal{C}_{p,e}}$*.*

Proposition 3.4. *For* $1 \leq i \leq 3$*, let* R_i *be a complete discrete valuation ring of characteristic* 0 *with perfect residue field of characteristic* p*. Let* $\mathfrak{m}_i$ *be the maximal ideal of* R_i *generated by* π_i *and* ν_i *corresponding valuation of* R_i*. For* $\phi^{1,2} : R_{1,n_1} \longrightarrow R_{2,n_2}$ *and* $\phi^{2,3} : R_{2,n_2} \longrightarrow R_{3,n_3}$*, suppose that there are liftings* $g^{1,2} : R_1 \longrightarrow R_2$ *and* $g^{2,3} : R_2 \longrightarrow R_3$ *of* $\phi^{1,2}$ *and* $\phi^{2,3}$ *respectively. If* $\nu_1(p) = \nu_2(p)$*, then* $g = g^{2,3} \circ g^{1,2}$ *is a lifting of* $\phi^{2,3} \circ \phi^{1,2}$*. Moreover* g *is a unique lifting of* $\phi^{2,3} \circ \phi^{1,2}$ *when* $n_3 > M(R_2)\nu_2(p)\nu_3(p)$ *and* $n_2 > M(R_1)\nu_1(p)\nu_2(p)$*.*

Proof. By Lemma 2.9, $M(R_1)$ is equal to $M(R_2)$, say M. Since $g^{1,2}$ is a lifting of $\phi^{1,2}$, there is a representative β_1 of $\phi^{1,2}(\pi_1 + \mathfrak{m}^{n_1}_1)$ such that $\widetilde{\nu_2}(g^{1,2}(\pi_1) - \beta_1) > M$. We note that β_1 is a uniformizer of R_2. Since $g^{2,3}$ is a lifting of $\phi^{2,3}$, there is a representative β_2 of $(\phi^{2,3} \circ \phi^{1,2})(\pi_1 + \mathfrak{m}^{n_1}_1) = \phi^{2,3}(\beta_1 + \mathfrak{m}^{n_2}_2)$ such that $\widetilde{\nu_3}(g^{2,3}(\beta_1) - \beta_2) > M$. If we write $g^{1,2}(\pi_1) = \beta_1 + x_M$ where $\widetilde{\nu_2}(x_M) > M$, then $g(\pi_1) = g^{2,3}(g^{1,2}(\pi_1)) = g^{2,3}(\beta_1 + x_M)$. Since $\widetilde{\nu_3}(g^{2,3}(\beta_1) - \beta_2) > M$ and $\widetilde{\nu_3}(g^{2,3}(x_M)) = \widetilde{\nu_2}(x_M) > M$,

$$\widetilde{\nu_3}\big(g(\pi_1) - \beta_2\big) = \widetilde{\nu_3}\big(g^{2,3}(\beta_1) - \beta_2 + g^{2,3}(x_M)\big) > M.$$

The equality $(\phi^{2,3} \circ \phi^{1,2})_{red,1} \circ \mathrm{pr}_{1,1} = \mathrm{pr}_{3,1} \circ g$ follows directly from $g = g^{2,3} \circ g^{1,2}$. By Definition 2.7 and Proposition 2.10, g is a lifting of $\phi^{2,3} \circ \phi^{1,2}$.

When $n_3 > M(R_2)\nu_2(p)\nu_3(p) = M(R_1)\nu_1(p)\nu_3(p)$ and $n_2 > M(R_1)\nu_1(p)\nu_2(p)$, g is a unique lifting of $\phi^{2,3} \circ \phi^{1,2}$ by Theorem 2.12.

□

Corollary 3.5. *Let* $e \geq 1$ *and* $R \in \mathrm{Ob}(\mathcal{C}_{p,e})$*. Suppose* $n > M(R)\nu(p)^2$*. Let* $\mathrm{pr}^{n,n}|_{\mathrm{Iso}(R)} : \mathrm{Iso}(R) \to \mathrm{Iso}(R_n)$ *be the natural projection map. Then there exists a surjective group homomorphism*

$$\mathrm{L}_n : \mathrm{Iso}(R_n) \longrightarrow \mathrm{Iso}(R)$$

which satisfies $\mathrm{L}_n \circ (\mathrm{pr}^{n,n}|_{\mathrm{Iso}(R)}) = \mathrm{Id}_{\mathrm{Iso}(R)}$*.*

Corollary 3.6. *Let R_1 be in $\mathrm{Ob}(\mathcal{C}_{p,e_1})$ and R_2 in $\mathrm{Ob}(\mathcal{C}_{p,e_2})$. Suppose $n_2 > M(R_1)\nu_1(p)\nu_2(p)$ and $\mathrm{Hom}(R_{1,n_1}, R_{2,n_2})$ is not empty. Then $\mathrm{Hom}(R_{1,n_1}, R_{2,n_2})$ and $\mathrm{Hom}(R_1, R_2)$ are right $\mathrm{Iso}(R_{1,n_1})$-sets and there exists a surjective $\mathrm{Iso}(R_{1,n_1})$-map*

$$\mathrm{L}_{n_1,n_2} : \mathrm{Hom}(R_{1,n_1}, R_{2,n_2}) \longrightarrow \mathrm{Hom}(R_1, R_2)$$

such that

$$\mathrm{L}_{n_1,n_2} \circ \mathrm{pr}^{n_1,n_2} = \mathrm{Id}_{\mathrm{Hom}(R_1,R_2)}$$

where $\mathrm{pr}^{n_1,n_2} : \mathrm{Hom}(R_1, R_2) \longrightarrow \mathrm{Hom}(R_{1,n_1}, R_{2,n_2})$ is the natural projection map.

Proof. It is clear that $\mathrm{Hom}(R_{1,n_1}, R_{2,n_2})$ is a right $\mathrm{Iso}(R_{1,n_1})$-set. By Lemma 2.5, we have $n_1 > M(R_1)\nu_1(p)^2$. Since $\mathrm{Hom}(R_1, R_2)$ is a right $\mathrm{Iso}(R_1)$-set, $\mathrm{Hom}(R_1, R_2)$ is a right $\mathrm{Iso}(R_{1,n_1})$-set via $\mathrm{L}_n : \mathrm{Iso}(R_n) \longrightarrow \mathrm{Iso}(R)$ by Corollary 3.5. Moreover, by Proposition 3.4, the lifting map L_{n_1,n_2} is a $\mathrm{Iso}(R_{1,n_1})$-map. □

Lemma 3.7. *Let $R \subset S$ be discrete valuation rings and S a finitely generated R-module. The discriminant $\mathcal{D}_{S/R}$ of S/R is equal to the norm $\mathrm{Norm}(\mathfrak{D}_{S/R})$ of the different $\mathfrak{D}_{S/R}$ of S/R.*

Proof. Chapter 3, Section 2 of [25]. □

Even though next corollary directly follows from Corollary 3.6, we state here because it is useful for numerical calculations.

Corollary 3.8. *Let R_1 and R_2 be complete discrete valuation rings of characteristic 0 with perfect residue fields of characteristic p. Let $\mathfrak{m}_i$ be the maximal ideal of R_i generated by π_i and ν_i corresponding valuation of R_i for $i = 1, 2$. Let L_i and K_i be the fraction fields of R_i and $W(k_i)$ for $i = 1, 2$ respectively. Suppose $\mathrm{Hom}(R_{1,n_1}, R_{2,n_2})$ is not empty. Then there is a surjective $\mathrm{Iso}(R_{1,n_1})$-map*

$$\mathrm{L}_{n_1,n_2} : \mathrm{Hom}(R_{1,n_1}, R_{2,n_2}) \longrightarrow \mathrm{Hom}(R_1, R_2)$$

such that

$$\mathrm{L}_{n_1,n_2} \circ \mathrm{pr}^{n_1,n_2} = \mathrm{Id}_{\mathrm{Hom}(R_1,R_2)}$$

if one of the following holds :

- *L_1/K_1 is a Galois extension.*
  i is the least number such that the i-th ramification group G_i of $Gal(L_1/K_1)$ vanishes.
  $n_2 > \nu_2(p)i$.
- *$n_2 > \nu_2(p) + \nu_2(p)\nu_1(\nu_1(p))$.*
- *$n_2 > \nu_2(\mathcal{D}_{R_1/W(k_1)})$ where $\mathcal{D}_{R_1/W(k_1)}$ is the discriminant of $R_1/W(k_1)$. Here $\nu_2(\mathcal{D}_{R_1/W(k_1)})$ means $\nu_2(p^a)$ where $\mathcal{D}_{R_1/W(k_1)} = p^a W(k_1)$.*

Proof.
- We recall that G_i is defined by $G_i = \{\sigma \in Gal(L_1/K_1) : \nu_1(\sigma(\pi_1) - \pi_1) \geq i + 1\}$. Then i is equal to $M(R_1)\nu_1(p)$.
- By Lemma 2.15, one can obtain

$$\begin{aligned} M(R_1) &\leq \widetilde{\nu_1}\big(f'(\alpha)\big) - \frac{\nu_1(p) - 2}{\nu_1(p)} \\ &\leq \frac{\nu_1(p) - 1 + \nu_1\big(\nu_1(p)\big)}{\nu_1(p)} - \frac{\nu_1(p) - 2}{\nu_1(p)} \\ &\leq \frac{1 + \nu_1\big(\nu_1(p)\big)}{\nu_1(p)} \end{aligned}$$

as in the proof of Theorem 2.16. Hence,

$$M(R_1)\nu_1(p)\nu_2(p) \leq \nu_2(p) + \nu_2(p)\nu_1\big(\nu_1(p)\big).$$

- Let $\mathfrak{D}_{R_1/W(k_1)}$ be the different of $R_1/W(k_1)$. Then one can obtain that

$$\begin{aligned}\nu_2(\mathcal{D}_{R_1/W(k_1)}) &= \nu_2(p)\widetilde{\nu_2}(\mathcal{D}_{R_1/W(k_1)})\\ &= \nu_2(p)\widetilde{\nu_1}(\mathcal{D}_{R_1/W(k_1)})\\ &= \nu_2(p)\nu_1(p)\widetilde{\nu_1}(\mathfrak{D}_{R_1/W(k_1)})\\ &= \nu_2(p)\nu_1(p)\widetilde{\nu_1}\big(f'(\pi_1)\big)\\ &\geq \nu_2(p)\nu_1(p)M(R_1).\end{aligned}$$

The second equality follows from the fact that $\widetilde{\nu_i}$ is normalized, the third equality follows from Lemma 3.7 and the fourth equality follows from Lemma 2.15 where f is the monic irreducible polynomial of π_1 over K_1.

□

Theorem 3.9. *The lifting number for $\mathcal{C}_{p,e}$ is finite. More precisely, $\mathcal{C}_{p,e}$ is $(e + e\nu(e) + 1)$-liftable. Here $\nu(e)$ denotes the exponent n such that e generates an ideal $\mathfrak{m}^n$ of R in $\mathrm{Ob}(\mathcal{C}_{p,e})$ where $\mathfrak{m}$ denotes the maximal ideal of R. $\nu(e)$ depends only on the prime number p and the ramification index e, in particular $\nu(e)$ is independent of the choice of R in $\mathrm{Ob}(\mathcal{C}_{p,e})$.*

Proof. Suppose n is bigger than $e + e\nu(e)$. For any $\overline{R}, \overline{R_1}$ and $\overline{R_2}$ in $\mathrm{Ob}(\mathcal{R}^n_{p,e})$, by Theorem 2.16, we define $\mathrm{L}_n(\overline{R})$ to be a unique ring R in $\mathrm{Ob}(\mathcal{C}_{p,e})$ which satisfies $\mathrm{Pr}_n(R) = \overline{R}$. As in the proof of Theorem 2.16, $e + e\nu(e) \geq M(R)e^2$. By Theorem 2.12, for any $\phi : \overline{R_1} \longrightarrow \overline{R_2}$, there exists a unique n-th lifting map $\mathrm{L}(\phi) : \mathrm{L}(\overline{R_1}) \longrightarrow \mathrm{L}(\overline{R_2})$, and hence we obtain a functor $\mathrm{L}_n : \mathcal{R}^n_{p,e} \longrightarrow \mathcal{C}_{p,e}$ by Proposition 3.4. By Definition 2.7, L_n is a lifting functor.

□

Remark 3.10. *For a fixed absolutely unramified valued field K, $M(L)e(L)$ can be arbitrarily large when extension degrees $[L : K]$ vary. For example, we can take $L = \mathbb{Q}_p(\zeta_{p^n})$ and $K = \mathbb{Q}_p$. More generally, if L runs through subfields of a deeply ramified extension of a local field K(see* [10] *for the definition of deeply ramified extensions), then $M(L)e(L)$ can be arbitrarily large. But* Lemma 2.15 *and the proof of* Theorem 2.16 *show that $M(L)e(L)$ must be bounded if we fix $[L : K]$. Hence we deduce the finiteness of the lifting number for $\mathcal{C}_{p,e}$.*

Example 2.13.(1) can be generalized as follows.

Proposition 3.11. *Let $R_1/W(k)$ and $R_2/W(k)$ be totally ramified extensions of degree e. Then $R_{1,e}$ is isomorphic to $R_{2,e}$ as $W(k)$-algebras.*

Proof. Let π_i be a uniformizer of R_i and ν_i the valuation corresponding to R_i for $i = 1, 2$. By the theory of totally ramified extensions(see Chapter 2 of [24] for example), the monic irreducible polynomial f_i of π_i over $W(k)$ is an Eisenstein polynomial for $i = 1, 2$. If we write $f_i = x^e + a_{i,e-1}x^{e-1} + ... + a_{i,1}x + a_{i,0}$, then

$\nu_i(p) = \nu_i(a_{i,0}) = e$ and $\nu_i(a_{i,j}) \geq e$ for $i = 1, 2$ and $j = 1, 2, ..., e-1$. This shows

$$\begin{aligned} R_{i,e} &= \frac{W(k)[\pi_i]}{(\pi_i)^e} \\ &\cong \frac{W(k)[x]}{(p, f_i)} \\ &= \frac{k[x]}{(x^e + ... + a_{i,1}x + a_{i,0})} \\ &= \frac{k[x]}{(x^e)}, \end{aligned}$$

and hence, $R_{1,e}$ is isomorphic to $R_{2,e}$ as $W(k)$-algebras.

□

Now we focus on tamely ramified extensions. For the tame case, we can calculate the lifting number.

Lemma 3.12. *For a perfect field k of characteristic p, let K be the fraction field of the Witt ring $W(k)$ of k. For a positive integer e prime to p, suppose that there is a prime divisor l of e such that ζ_{l^n} is in $k^\times$ and $\zeta_{l^{n+1}}$ is not in $k^\times$ for some n. Then there are two totally ramified extensions L_1 and L_2 of degree e over K which are not isomorphic over $\mathbb{Q}$.*

Proof. ζ_{l^n} is in $W(k)^\times$ and $\zeta_{l^{n+1}}$ is not in $W(k)^\times$ by Hensel's lemma. Then $L_1 = K(\sqrt[e]{p})$ and $L_2 = K(\sqrt[e]{p\zeta_{l^n}})$ are totally ramified extensions of degree e over K. Suppose that there is an isomorphism $\sigma : L_2 \longrightarrow L_1$. Since Galois conjugates of $\sqrt[e]{p}$ and ζ_{el^n} over $\mathbb{Q}$ are $\sqrt[e]{p}\zeta_e^i$ and $\zeta_{el^n}^j$ for each i and j where j is prime to e respectively,

$$\sigma\left(\sqrt[e]{p\zeta_{l^n}}\right) = \sigma\left(\sqrt[e]{p}\zeta_{el^n}\right) = \sqrt[e]{p}\zeta_{el^n}^k$$

for some k prime to l. In particular, L_1 contains both $\sqrt[e]{p}$ and $\sqrt[e]{p}\zeta_{el^n}^k$, and hence, $\zeta_{l^{n+1}}$ is in L_1. This is a contradiction since L_1/K is totally ramified.

□

Corollary 3.13. *Suppose that p does not divide e and $e > 1$. Then $e+1$ is the lifting number for $\mathcal{C}_{p,e}$.*

Proof. Since $\nu(p) = 0$, $e + e\nu(e) + 1 = e + 1$. By Theorem 3.9, $\mathcal{C}_{p,e}$ is $(e+1)$-liftable. Let $\mathbb{F}_p$ be the prime field of p elements. Let K be the fraction field of the Witt ring $W(k)$ of $k = \mathbb{F}_p(\zeta_e)$. By Lemma 3.12, there are two totally ramified extensions L_1 and L_2 of degree e over K such that there is no isomorphism between L_1 and L_2. If $\mathcal{C}_{p,e}$ is e-liftable, L_1 and L_2 are isomorphic over K by Proposition 3.11 and it is a contradiction. □

Remark 3.14. Proposition 3.11 *and* Corollary 3.13 *show the difference between the unramified case and the tamely ramified case. We can regard the absolutely unramified valued fields of mixed characteristic as the absolutely tamely ramified valued fields having the ramification index $e = 1$. If we apply the formula $e+1$ in* Corollary 3.13 *to $\mathcal{C}_p$, the lifting number for $\mathcal{C}_p$ should be $1+1 = 2$. But the argument in the proof of* Corollary 3.13 *does not work for $\mathcal{C}_p$. For an absolutely unramified complete discrete valued field K, there is a unique totally ramified extension of* *degree* 1 *over* K*, that is,* K *itself. Hence the fact that the lifting number for* $\mathcal{C}_p$ *is* 1 *does not disagree with* Corollary 3.13*.*

For the wild case, we have the following example. Let $R_1 = \mathbb{Z}_2[\sqrt{2}]$ and $R_2 = \mathbb{Z}_2[\sqrt{10}]$. There is no homomorphism between R_1 and R_2 by Kummer theory. But there is an isomorphism between $R_{1,6}$ and $R_{2,6}$ since

$$\begin{aligned} R_{1,6} = \frac{\mathbb{Z}_2[\sqrt{2}]}{(\sqrt{2}^6)} &\cong \frac{\mathbb{Z}_2[x]}{(x^2-2,8)} \\ = \frac{\mathbb{Z}_2[x]}{(x^2-10,8)} &\cong \frac{\mathbb{Z}_2[\sqrt{10}]}{(\sqrt{2}^6)} = R_{2,6}. \end{aligned}$$

This shows that the lifting number for $\mathcal{C}_{2,2}$ is $2+2\nu(2)+1 = 7 > \nu(2)$ by Theorem 3.9. In general, we have the lower bound $e+1$ of the lifting number for the wild case. For proving this, we need the following lemma.

Lemma 3.15. *For a perfect field* k *of characteristic* p*, let* K *be the fraction field of the Witt ring* $W(k)$ *of* k*. Let* e *be a positive integer divided by* p*. Then there are two totally ramified extensions* L_1 *and* L_2 *of degree* e *over* K *which are not isomorphic over* $\mathbb{Q}$*.*

Proof. We write $e = sp^r$ for some positive integers s and r where s is prime to p. Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension, in particular $Gal(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p$. Let M_r be a unique subfield of $\mathbb{Q}_\infty$ such that $[M_r : \mathbb{Q}] = p^r$. By the theory of cyclotomic fields(See for example Chapter 1 of [25]), the Galois extension $M_r/\mathbb{Q}$ is totally ramified at the place above p. Let α be a uniformizer of M_r corresponding to the place above p. Since, $M_r/\mathbb{Q}$ is a Galois extension, $M_r = \mathbb{Q}(\alpha) = \mathbb{Q}(\sigma(\alpha))$ for any embedding σ. We fix an embedding $\mathbb{Q}^{alg} \subset K^{alg}$.

Let $L_1 = K(p^{1/e}) = K(p^{1/s}, p^{1/p^r})$ and $L_2 = K(p^{1/s}, \alpha)$. Then L_1 and L_2 are totally ramified extensions of degree e over K. If there is an isomorphism $\sigma : L_2 \longrightarrow L_1$, L_1 contains both $\sigma(\alpha)$ and p^{1/p^r}. Since $\mathbb{Q}(\alpha) = \mathbb{Q}(\sigma(\alpha))$, $K(\sigma(\alpha)) = K(\alpha)$ is contained in L_1. We note that $[K(p^{1/p^r}, \alpha) : K(p^{1/p^r})]$ divides $[K(\alpha) : K] = p^r$ since $K(\alpha)/K$ is a Galois extension. Since

$$s = \left[L_1 : K\left(p^{1/p^r}\right)\right] = \left[L_1 : K\left(p^{1/p^r}, \alpha\right)\right]\left[K\left(p^{1/p^r}, \alpha\right) : K(p^{1/p^r})\right],$$

$[K(p^{1/p^r}, \alpha) : K(p^{1/p^r})]$ divides s. Hence we obtain $[K(p^{1/p^r}, \alpha) : K(p^{1/p^r})] = \gcd(s, p^r) = 1$. This shows $K(p^{1/p^r}) = K(\alpha)$ since $[K(p^{1/p^r}) : K] = [K(\alpha) : K]$. This is a contradiction, and hence, L_1 and L_2 are not isomorphic. □

Proposition 3.16. *Let* p *be a prime number and* e *be a natural number divided by* p*. Then the lifting number for* $\mathcal{C}_{p,e}$ *is bigger than* e*.*

Proof. By Lemma 3.15, there are two totally ramified extensions L_1 and L_2 of degree e over $\mathbb{Q}_p$ such that there is no isomorphism over $\mathbb{Q}_p$ between L_1 and L_2. If $\mathcal{C}_{p,e}$ is e-liftable, L_1 and L_2 are isomorphic over $\mathbb{Q}_p$ by Proposition 3.11 and it is a contradiction. Hence, the lifting number for $\mathcal{C}_{p,e}$ is bigger than e.

□

Corollary 3.17. *The lifting number for* $\mathcal{C}_{p,e}$ *is bigger than* e *whenever* $e > 1$*.*

Although we have the lower bound $e+1$ and the upper bound $e+e\nu(e)+1$ of the lifting number for $\mathcal{C}_{p,e}$, we have no clue to calculate the lifting number explicitly for the wild case.

Question 3.18. *What is the lifting number for the wild case ?*

Remark 3.19. *When $e>1$, for a lifting functor $\mathrm{L}_n : \mathcal{R}_{p,e}^n \longrightarrow \mathcal{C}_{p,e}$ and $\overline{R}$ in $\mathcal{R}_{p,e}^n$, any complete discrete valuation ring R which has $\overline{R}$ as its n-th residue ring necessarily has ramification index e and equal to $\mathrm{L}_n(\overline{R})$ by* Corollary 3.17*. But for the lifting functor $\mathrm{L} : \mathcal{R}_p \longrightarrow \mathcal{C}_p$ in* Theorem 3.1*, there is no information on ramification indices in $\mathrm{Ob}(\mathcal{R}_p)$. Really there are many complete discrete valuation rings with different ramification indices which have the same residue field. For example, $R_1 = \mathbb{Z}_3[\sqrt[3]{3}]$ and $R_2 = \mathbb{Z}_3[\sqrt[2]{3}]$ have the same residue field, but their ramification indices are different.*

For a fixed set $\mathcal{N} = \{n_e \in \mathbb{N}\}_{e\in\mathbb{N}}$, if we try to make a unified lifting functor from $\mathcal{R}_{\mathcal{N}} := \bigcup_e \mathcal{R}_{p,e}^{n_e}$ to $\mathcal{C} := \bigcup_e \mathcal{C}_{p,e}$, we can not apply our method to get such a functor since $M(R_e)e$ is unbounded as varying e and $R_e \in \mathrm{Ob}(C_{p,e})$. But we have a unified functor for a finite set of ramification indices.

Corollary 3.20. *For a finite set $\{e_1, ..., e_s\}$ of ramification indices, there is a finite set of natural numbers $\{n_1, .., n_s\}$ such that there exists a lifting functor*

$$\mathrm{L} : \bigcup_{1\le k\le s} \mathcal{R}_{p,e_k}^{n_k} \longrightarrow \bigcup_{1\le k\le s} \mathcal{C}_{p,e_k}^{n_k}.$$

Proof. Follows from Lemma 2.5, Proposition 3.4, Corollary 3.8 and Theorem 3.9. □

4. Ax-Kochen-Ershov principle for finitely ramified valued fields

Our main goal in this section is to prove a strengthened version of Basarab's result on the AKE-principle for finitely ramified valued fields in the case of perfect residue fields. Firstly, we quickly review the basic results in model theory of valued fields, concentrating on the AKE-principle. We take the language of valued fields, which consists of three types of sorts for valuation fields, residue fields, and value groups. Let $\mathcal{L}_K = \{+,-,\cdot;0,1\}$ be the ring language for valued fields, $\mathcal{L}_k = \{+',-',\cdot';0',1'\}$ be the ring language for residue fields, and $\mathcal{L}_\Gamma = \{+^*;0^*;<\}$ be the ordered group language for value groups. Let $\mathcal{L}_{val} = \mathcal{L}_K \cup \mathcal{L}_k \cup \mathcal{L}_\Gamma$ be the language of valued fields. Next, we consider an extended language of $\mathcal{L}_{val}$ by adding the ring languages for the n-th residue rings. For each $n \le 1$, let $\mathcal{L}_{R_n} = \{+_n, -_n, \cdot_n; 0_n, 1_n\}$ be the ring language for the n-th residue ring. For $n=1$, we identify $\mathcal{L}_{R_1} = \mathcal{L}_k$. We get an extended language $\mathcal{L}_{val,R} = \mathcal{L}_{val} \cup \bigcup_{n\ge1} \mathcal{L}_{R_n}$ for valued fields. Let (K_1,ν_1,k_1,Γ_1) and (K_2,ν_2,k_2,Γ_2) be valued fields, and let $R_{1,n}$ and $R_{2,n}$ be the n-th residue rings of (K_1,ν_1) and (K_2,ν_2) respectively. We say (K_1,ν_1) and (K_2,ν_2) are elementarily equivalent if they are elementarily equivalent in $\mathcal{L}_{val}$. If (K_1,ν_1) and (K_2,ν_2) are elementarily equivalent, then they are elementarily equivalent in $\mathcal{L}_{val,R}$ because the n-th residue rings are interpretable in $\mathcal{L}_{val}$. For (K_1,ν_1) and (K_2,ν_2) which are elementarily equivalent, it necessarily implies that

- k_1 and k_2 are elementarily equivalent in $\mathcal{L}_k$;
- Γ_1 and Γ_2 are elementarily equivalent in $\mathcal{L}_\Gamma$; and
- $R_{1,n}$ and $R_{2,n}$ are elementarily equivalent in $\mathcal{L}_{R_n}$ for each $n \le 1$.

Ax and Kochen in [4], and Ershov in [14] proved the fact that these conditions on the residue fields and the value groups imply elementary equivalence for unramified valued fields:

Theorem 4.1. [4, 14]*(The Ax-Kochen-Ershov principle) Let* $(K_1, \nu_1, k_1, \Gamma_1)$ *and* $(K_2, \nu_2, k_2, \Gamma_2)$ *be unramified henselian valued fields of characteristic zero.*

$$K_1 \equiv K_2 \text{ if and only if } k_1 \equiv k_2 \text{ and } \Gamma_1 \equiv \Gamma_2.$$

Basarab in [6] extended Theorem 4.1 for henselian valued fields of finite ramification indices, including local fields of characteristic zero.

Theorem 4.2. [6] *Let* $(K_1, \nu_1, k_1, \Gamma_1)$ *and* $(K_2, \nu_2, k_2, \Gamma_2)$ *be henselian valued fields of mixed characteristic having finite absolute ramification indices. The following are equivalent :*

(1) $K_1 \equiv K_2$.
(2) $R_{1,n} \equiv R_{2,n}$ *for each* $n \leq 1$ *and* $\Gamma_1 \equiv \Gamma_2$.

Next we review on the coarse valuations. For the coarse valuations, we refer to [13, 20, 27, 30].

Remark/Definition 4.3. [27] *Suppose* (K, ν, k, Γ) *has the finite absolute ramification index so that the value group has the minimum positive element, and let* π *be a uniformizer so that* $\nu(\pi)$ *is the smallest positive element in* Γ. *Let* Γ° *be the convex subgroup of* Γ *generated by* $\nu(\pi)$ *and* $\dot{\nu} : K \setminus \{0\} \longrightarrow \Gamma/\Gamma^\circ$ *be a map sending* $x(\neq 0) \in K$ *to* $\nu(x) + \Gamma^\circ \in \Gamma/\Gamma^\circ$. *The map* $\dot{\nu}$ *is a valuation, called* the coarse valuation. *The residue field* K°, *called* the core field *of* (K, ν), *of* $(K, \dot{\nu})$ *forms a valued field equipped with a valuation* ν° *induced from* ν *and the value groups* Γ°. *More precisely, the valuation* ν° *is defined as follows: Let* $\mathrm{pr}_{\dot{\nu}} : R_{\dot{\nu}} \longrightarrow K^\circ$ *be the canonical projection map and let* $x \in R_{\dot{\nu}}$. *If* $x^\circ := \mathrm{pr}_{\dot{\nu}}(x) \in K^\circ \setminus \{0\}$, *then* $\nu^\circ(x^\circ) := \nu(x)$. *And* $x^\circ = 0 \in K^\circ$ *if and only if* $\nu(x) > \gamma$ *for all* $\gamma \in \Gamma^\circ$.

Lemma 4.4. *(1) Let* R_ν, $R_{\dot{\nu}}$, *and* R_{ν° *be the valuation rings of* (K, ν), $(K, \dot{\nu})$, *and* (K°, ν°) *respectively. Then* $(\mathrm{pr}_{\dot{\nu}})^{-1}(R_{\nu^\circ}) = R_\nu$.

(2) *Let* R_n *and* R_n° *be the* n*-th residue rings of* (K, ν) *and* (K°, ν°) *respectively. Then there is a canonical isomorphism* $\theta_n : R_n \longrightarrow R_n^\circ$ *such that* $\mathrm{pr}_n^{\nu^\circ} \circ (\mathrm{pr}_{\dot{\nu}} |_{R_\nu}) = \theta_n \circ \mathrm{pr}_n$, *where* $\mathrm{pr}_n : R_\nu \longrightarrow R_n$ *and* $\mathrm{pr}_n^{\nu^\circ} : R_{\nu^\circ} \longrightarrow R_n^\circ$ *are the canonical projection map.*
(3) *If* (K, ν) *is henselian, then* $(K, \dot{\nu})$ *is henselian.*
(4) *If* (K, ν) *is* $\aleph_1$*-saturated, then* (K°, ν°) *is complete.*

Proof. (1) Note that $R_{\dot{\nu}} := \{x \in K |\ \dot{\nu}(x) \geq 0\} = \{x \in K |\ \nu(x) \geq \gamma$ for some $\gamma \in \Gamma^\circ\}$. Let $x \in R_{\dot{\nu}}$ be such that $\mathrm{pr}_{\dot{\nu}}(x) =: x^\circ \in R_{\nu^\circ}$, that is, $\nu^\circ(x^\circ)(\in \Gamma^\circ) \geq 0$. If $x^\circ = 0$, $\nu(x) > \gamma$ for all $\gamma \in \Gamma^\circ$ and $x \in R_\nu$. If $x^\circ \neq 0$, then $\nu^\circ(x^\circ) = \nu(x) \geq 0$ in Γ° ,and hence $\nu(x) \geq 0$ in Γ. Thus $x \in R_\nu$. Therefore, for $x \in R_{\dot{\nu}}$, $x \in R_\nu$ if and only if $x^\circ \in R_{\nu^\circ}$.

(2) Note that each θ_n is induced from $\mathrm{pr}_{\dot{\nu}} |_{R_\nu} : R_\nu \longrightarrow R_{\nu^\circ}$. It is easy to see that each θ_n is surjective. To show that θ_n is injective, it is enough to show that $\nu(x) \geq n$ if and only if $\nu^\circ(x^\circ) \geq n$ for $x \in R_\nu$. It clearly comes from the definition of ν° in (1).

(3)-(4) Section 5 of [20]. □

Proposition 4.5. *Let* (K_1, ν_1, Γ_1) *and* (K_2, ν_2, Γ_2) *be valued fields. Let* $R_{1,n}$ *and* $R_{2,n}$ *be the* n*-th residue rings of* K_1 *and* K_2 *respectively. Suppose*

- *$R_{1,n} \equiv R_{2,n}$ for each $n \geq 1$;*
- *$\Gamma_1 \equiv \Gamma_2$*

Then there are $\aleph_1$-saturated elementary extensions $(K_1', \nu_1', \Gamma_1')$ and $(K_2', \nu_2', \Gamma_2')$ of K_1 and K_2 such that

- *$R_{1,n}' \cong R_{2,n}'$ for $n \geq 1$;*
- *$\Gamma_1' \cong \Gamma_2'$*

, where $R_{1,n}'$ and $R_{2,n}'$ are the n-th residue rings of K_1' and K_2' respectively.

Proof. We inductively construct chains of valued fields $(K_1^i, \Gamma_1^i)_{i\in\omega}$ and $(K_2^i, \Gamma_2^i)_{i\in\omega}$, and isomorphisms $\xi_j^i : R_{1,j}^i \longrightarrow R_{2,j}^i$ for $0 < i$ and $1 \leq j \leq i$, where $R_{1,j}^i$ and $R_{2,j}^i$ are the j-th residue rings of K_1^i and K_2^i respectively such that for $i \in \omega$,

$(1)_i$ $K_1^i \prec K_1^{i+1}$ and $K_2^i \prec K_2^{i+1}$;
$(2)_i$ $\xi_j^i \subset \xi_j^{i+1}$ for $1 \leq j \leq i$;
$(3)_i$ $\Gamma_1^i \cong \Gamma_2^i$.

Recall the Keisler-Shelah isomorphism theorem :

Theorem 4.6. *(Keisler-Shelah Isomorphism Theorem) Let $\mathcal{M}$ and $\mathcal{N}$ be two first order structures. If $\mathcal{M} \equiv \mathcal{N}$, then there is a ultrafilter $\mathcal{U}$ on an infinite set I such that*

$$\mathcal{M}^{\mathcal{U}} \cong \mathcal{N}^{\mathcal{U}},$$

where $\mathcal{M}^{\mathcal{U}}$ and $\mathcal{N}^{\mathcal{U}}$ are the ultrapowers of $\mathcal{M}$ and $\mathcal{N}$ with respect to $\mathcal{U}$.

Proof. See [11]. □

Since $\Gamma_1 \equiv \Gamma_2$, by Theorem 4.6, there is an ultrafilter $\mathcal{U}_0$ such that $\Gamma_1^{\mathcal{U}_0} \cong \Gamma_2^{\mathcal{U}_0}$. Set $(K_1^0, \Gamma_1^0) = (K_1^{\mathcal{U}_0}, \Gamma_1^{\mathcal{U}_0})$ and $(K_2^0, \Gamma_2^0) = (K_2^{\mathcal{U}_0}, \Gamma_2^{\mathcal{U}_0})$. Assume we construct sequences of valued fields $(K_1^i, \Gamma_1^i)_{i\leq m}$ and $(K_2^i, \Gamma_2^i)_{i\leq m}$ with isomorphisms $\xi_j^i : R_{1,j}^i \longrightarrow R_{2,j}^i$ for $1 \leq j \leq i \leq m$ for some $m \geq 0$ satisfying the conditions $(1)_i$, $(2)_i$, and $(3)_i$ for $i \leq m$. Since $R_{1,m+1}^m \equiv R_{2,m+1}^m$, from Theorem 4.6, there is an ultrafilter $\mathcal{U}$ such that $\xi_{m+1}^{m+1} : (R_{1,m+1}^m)^{\mathcal{U}} \cong (R_{2,m+1}^m)^{\mathcal{U}}$. Set $K_1^{m+1} = (K_1^m)^{\mathcal{U}}$ and $K_2^{m+1} = (K_2^m)^{\mathcal{U}}$, and set $\xi_j^{m+1} = (\xi_j^m)^{\mathcal{U}} : R_{1,j}^{m+1} \longrightarrow R_{2,j}^{m+1}$ for each $j \leq m$, where $R_{1,j}^{m+1} = (R_{1,j}^m)^{\mathcal{U}}$ and $R_{2,j}^{m+1} = (R_{2,j}^m)^{\mathcal{U}}$. Set $\Gamma_1^{m+1} = (\Gamma_1^m)^{\mathcal{U}} \cong (\Gamma_2^m)^{\mathcal{U}} = \Gamma_2^{m+1}$. Then the sequences of valued fields $(K_1^i, \Gamma_1^i)_{i\leq m+1}$ and $(K_2^i, \Gamma_2^i)_{i\leq m+1}$ with isomorphisms $\xi_j^i : R_{1,j}^i \longrightarrow R_{2,j}^i$ for $1 \leq j \leq i \leq m+1$ satisfying $(1)_i$, $(2)_i$, and $(3)_i$ for $i \leq m+1$. By induction, we get chains of valued fields, $(K_1^i)_{i\geq 0}$ and $(K_2^i)_{i\geq 0}$ with isomorphisms $\xi_n^i : R_{1,n}^i \longrightarrow R_{2,n}^i$ for $1 \leq n \leq i$ satisfying $(1)_i$, $(2)_i$, and $(3)_i$ for $i \geq 0$.

Next consider the unions $K_1^\omega := \bigcup K_1^i$ and $K_2^\omega := \bigcup K_2^i$, which are elementary extensions of K_1 and K_2 respectively. Each n-th residue ring $R_{k,n}^\omega$ is the union of $R_{k,n}^i$'s for $k = 1, 2$. For each $n \geq 1$, define $\xi_n^\omega := \bigcup \xi_n^i : R_{1,n}' \longrightarrow R_{2,n}'$. By $(2)_i$, the map ξ_n^ω is a well-defined isomorphism from $R_{1,n}^\omega$ to $R_{2,n}^\omega$. Hence K_1^ω and K_2^ω have the isomorphic n-th residue rings for each n. At last to get $\aleph_1$-saturated valued fields, consider an ultrapower $K_k^{\omega+1} := (K_k^\omega)^{\mathcal{U}'}$ with respect to a nonprincipal $\mathcal{U}'$ on ω for $k = 1, 2$, and $K_1^{\omega+1}$ and $K_2^{\omega+1}$ are desired valued fields. □

By combining Theorem 4.1 and Lemma 4.4, Proposition 4.5, we reduce the problem on elementary equivalence between henselian valued fields of mixed characteristic having finite ramification indices to the problem on isometricity between complete discrete valued fields of mixed characteristic whose the n-th residue rings are isomorphic for each $n \geq 1$. Now we improve Theorem 4.2.

Theorem 4.7. *Let $(K_1, \nu_1, k_1, \Gamma_1)$ and $(K_2, \nu_2, k_2, \Gamma_1)$ be henselian valued fields of mixed characteristic with finite ramification indices. Suppose k_1 and k_2 are perfect fields of characteristic $p > 0$. For $n \geq 1$, let $R_{1,n}$ and $R_{2,n}$ be the n-th residue rings of K_1 and K_2 respectively. Let $n_0 > \max\{e_{\nu_1}(p)(1 + e_{\nu_1}(e_{\nu_1}(p)), e_{\nu_2}(p)(1 + e_{\nu_2}(e_{\nu_2}(p))\}$. The following are equivalent:*

(1) $K_1 \equiv K_2$;

(2) $\Gamma_1 \equiv \Gamma_2$ and $R_{1,n} \equiv R_{2,n}$ for each $n \leq 1$; and

(3) $\Gamma_1 \equiv \Gamma_2$ and $R_{1,n_0} \equiv R_{2,n_0}$.

Proof. Let $(K_1, \nu_1, k_1, \Gamma_1)$ and $(K_2, \nu_2, k_2, \Gamma_2)$ be henselian valued fields of characteristic zero with the finite ramification indices so that Γ_1 and Γ_2 have the minimum positive elements. Suppose k_1 and k_2 are perfect fields of characteristic $p > 0$. It is easy to check (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We show (3) $\Rightarrow$ (1).

(3) $\Rightarrow$ (1). Suppose $R_{1,n_0} \equiv R_{2,n_0}$ and $\Gamma_1 \equiv \Gamma_2$. By the proof of Proposition 4.5, we may assume that $R_{1,n_0} \cong R_{2,n_0}$ and $\Gamma_1 \cong \Gamma_2$, and that (K_1, ν_1, Γ_1) and (K_2, ν_2, Γ_2) are $\aleph_1$-saturated. Consider the coarse valuations $\dot{\nu}_1$ and $\dot{\nu}_2$ of ν_1 and ν_2 respectively and the valued fields $(K_1, \dot{\nu}_1, \Gamma_1/\Gamma_1^\circ)$ and $(K_2, \dot{\nu}_2, \Gamma_2/\Gamma_2^\circ)$, where Γ_i° is the convex subgroup of Γ_i generated by the minimum positive element in Γ_i for $i = 1, 2$. Since (K_1, ν_1) and (K_2, ν_2) are $\aleph_1$-saturated, by Lemma 4.4.(4), the core fields (K_1°, ν_1°) and (K_2°, ν_2°) are complete discrete valued fields, where ν_1° and ν_2° are the valuation induced from ν_1 and ν_2 respectively. Since the n_0-th residue rings of (K_1, ν_1) and (K_2, ν_2) are isomorphic, by Lemma 4.4.(2), the n_0-th residue rings of (K_1°, ν_1°) and (K_2°, ν_2°) are isomorphic.

By Theorem 2.12 and the proof of Theorem 2.16, K_1° and K_2° are isomorphic. Since $\Gamma_1 \cong \Gamma_2$, $\Gamma_1/\Gamma_1^\circ \cong \Gamma_2/\Gamma_2^\circ$. Therefore by Theorem 4.1, $(K_1, \dot{\nu}_1) \equiv (K_2, \dot{\nu}_2)$. To get that $(K_1, \nu_1) \equiv (K_2, \nu_2)$, it is enough to show that the valuation rings R_{ν_1} of (K_1, ν_1) and R_{ν_2} of (K_2, ν_2) are definable in $(K_1, \dot{\nu}_1)$ and $(K_2, \dot{\nu}_2)$ by the same formula. We need the following lemma on a definability of valuation ring in the ring language.

Lemma 4.8. *Let (K, ν) be a complete field of characteristic zero. Suppose the residue field k is perfect and has prime characteristic p. Then the valuation ring R_ν of (K, ν) is definable by the formula*

$$\phi_q(x) \equiv \exists y\ y^q = 1 + px^q$$

for some $q > 0$ such that $p \not| q$ and $q > e_\nu(p)$. For example, we can take q as $p^l + 1$ for sufficiently large $l > 0$.

Proof. See [8]. □

Take $l > 0$ large enough so that $q := p^l + 1 > \max\{e_{\nu_1}(p), e_{\nu_2}(p)\}$. By Lemma 4.8, $\phi_q(x)$ defines the residue rings $R_{\nu_1^\circ}$ and $R_{\nu_2^\circ}$ of (K_1°, ν_1°) and (K_2°, ν_2°). By Lemma 4.4.(1), the valuation rings R_{ν_1} and R_{ν_2} are definable by the same formula in $(K_1, \dot{\nu}_1)$ and $(K_2, \dot{\nu}_1)$ so that $(K_1, \nu_1) \equiv (K_2, \nu_2)$. □

We give some corollaries of Theorem 4.7. At first, we improve the result in [5] on a decidability of henselian valued fields of finite absolute ramification indices in the case of perfect residue fields.

Corollary 4.9. *Let (K, ν, Γ) be a henselian valued field of mixed characteristic having finite absolute ramification index and the perfect residue field. Let R_n be the n-th residue ring of (K, ν) for each $n \geq 1$. Let $n_0 > e_\nu(p)(1 + e_\nu(e_\nu(p))$. Let*

$\mathrm{Th}(K,\nu)$ be the theory of (K,ν), $\mathrm{Th}(\Gamma)$ be the theory of Γ, and $\mathrm{Th}(R_n)$ be the theory of R_n. The following are equivalent :

(1) $\mathrm{Th}(K,\nu)$ *is decidable.*
(2) $\mathrm{Th}(\Gamma)$ *is decidable, and* $\mathrm{Th}(R_n)$ *is decidable for each* $n \geq 1$.
(3) $\mathrm{Th}(\Gamma)$ *is decidable, and* $\mathrm{Th}(R_{n_0})$ *is decidable.*

Proof. (1) $\Leftrightarrow$ (2) It was already given by Basarab in [5].

(1) $\Leftrightarrow$ (3) Let (K,ν,Γ,k) be a henselian valued fields of mixed characteristic having a perfect residue field k. Let $p>0$ be the characteristic of k and let $e(:= e_\nu(p))$ be the absolute ramification index of (K,ν). Suppose e is finite. Consider the following theory $T_{p,e}$ consisting of the following statements, which can be expressed by the first order logic;

- (K,ν) is a henselian valued field of characteristic zero;
- Γ is an abelian ordered group having the minimum positive element;
- k is a perfect field of characteristic $p>0$;
- (K,ν) has the absolute ramification index e.

By Theorem 4.7, the theory $T_{p,e}\cup \mathrm{Th}(\Gamma)\cup \mathrm{Th}(R_{n_0})$ is complete. Thus $\mathrm{Th}(K,\nu)$ is decidable if and only if $\mathrm{Th}(\Gamma)$ and $\mathrm{Th}(R_{n_0})$ are decidable. □

Thus we get the following results on local fields of mixed characteristic.

Corollary 4.10. [5][6] *Let (K_1,ν_1) and (K_2,ν_2) be local fields of mixed characteristic.*

(1) $(K_1,\nu_1)\equiv(K_2,\nu_2)\Leftrightarrow f: K_1\cong K_2$.
(2) $\mathrm{Th}(K_1,\nu_1)$ *is decidable.*

Next we recall the following definition introduced in [6]:

Definition 4.11. [6] *Let T be the theory of a henselian valued field (K,ν,Γ) of mixed characteristic having finite absolute ramification index e. Let $\lambda(T)\in\mathbb{N}\cup\{\infty\}$ be defined as the smallest positive integer n (if such a number exists) such that for every henselian valued field (K',ν',Γ') of mixed characteristic having the same absolute ramification index as (K,ν,Γ), the following are equivalent:*

(1) $(K',\nu',\Gamma')\models T$.
(2) $\Gamma\equiv\Gamma'$ *and the n-th residue rings of (K,ν) and (K',ν') are elementarily equivalent.*

Otherwise, $\lambda(T)=\infty$.

Question 4.12. [6] *Let T be the theory of a henslian valued field of mixed characteristic having finite absolute ramification index. Is $\lambda(T)$ finite ?*

It was proved that $\lambda(T)<\infty$ for the theories T of local fields of mixed characteristic in [6](but the statement and its proof are incorrect as we remarked in Section 2). We give a positive answer when residue fields are perfect.

Corollary 4.13. *Let (K,ν) be a henselian valued field of mixed characteristic having finite absolute ramification index with perfect residue field. Let T be the theory of (K,ν). Then $\lambda(T)\leq e_\nu(p)(1+e_\nu(e_\nu(p)))+1$.*

We compute explicitly $\lambda(T)$ for the theories T of some tamely ramified valued fields. We say that an abelian group G is e-divisible (respectively, uniquely e-divisible) when the multiplication by e map, $e: G\longrightarrow G$ is surjective (respectively, bijective). We denote the unit group of a ring R by $R^\times$.

Lemma 4.14. *Let $(K, W(k), \mathfrak{m}, k)$ be an absolutely unramified complete discrete valued field of mixed characteristic $(0,p)$ with perfect residue field k. Suppose that $k^\times$ is e-divisible for a positive integer e prime to p.*

(1) *If ζ_e is contained in $W(k)$, then there exists a unique totally tamely ramified extension L of degree e over K.*
(2) *If ζ_e is not contained in $W(k)$, then there exists a unique totally tamely ramified extension L of degree e over K up to K-isomorphism.*

Proof. Let S' be the group of nonzero Teichmüller representatives of $W(k)$ and $U^{(n)} = 1 + \mathfrak{m}^n$ the n-th principal unit group of $W(k)$ for each $n \geq 1$. Since

$$U^{(n)} = \ker\left(W(k)^\times \longrightarrow \left(\frac{W(k)}{\mathfrak{m}^n}\right)^\times\right)$$

and

$$W(k) = \varprojlim_n \left(\frac{W(k)}{\mathfrak{m}^n}\right),$$

we have

$$W(k)^\times = \varprojlim_n \left(\frac{W(k)}{\mathfrak{m}^n}\right)^\times = \varprojlim_n \left(\frac{W(k)^\times}{U^{(n)}}\right).$$

Since $W(k)^\times = S' \times U^{(1)}$, we obtain

$$U^{(1)} = \varprojlim_n \left(\frac{U^{(1)}}{U^{(n)}}\right).$$

Since $U^{(n)}/U^{(n+1)} \cong k$ for each $n \geq 1$, a short exact sequence

$$0 \longrightarrow \frac{U^{(n+1)}}{U^{(n+2)}} \longrightarrow \frac{U^{(n)}}{U^{(n+2)}} \longrightarrow \frac{U^{(n)}}{U^{(n+1)}} \longrightarrow 0$$

shows that $U^{(1)}/U^{(n)}$ is a p-group, and hence, uniquely e-divisible for each n. Hence, $U^{(1)}$ is uniquely e-divisible and $W(k)^\times = S' \times U^{(1)}$ is e-divisible since $k^\times \cong S'$.

(1) Suppose that ζ_e is contained in S'. Then there is a unique totally tamely ramified extension of degree e over K by Kummer theory since

$$\frac{K^\times}{(K^\times)^e} = \frac{p^{\mathbb{Z}} \times W(k)^\times}{p^{e\mathbb{Z}} \times (W(k)^\times)^e} \cong \frac{\mathbb{Z}}{e\mathbb{Z}}.$$

(2) Suppose that ζ_e is not contained in S'. For a totally tamely ramified extension L of degree e over K, there is u in $W(k)^\times$ such that $L = K(\sqrt[e]{pu})$ by the theory of tamely ramified extensions(see Chapter 2 of [24] for example). Since $W(k)^\times$ is e-divisible, there is v in $W(k)^\times$ such that $v^e = u$. Hence, $\sqrt[e]{pu} = \sqrt[e]{p}v\zeta_e^i$ for some i. This shows that $L = K(\sqrt[e]{pu}) = K(\sqrt[e]{p}\zeta_e^i)$ is isomorphic to $K(\sqrt[e]{p})$ over K since the irreducible polynomial of $\sqrt[e]{p}$ over K is $x^e - p$.

□

Proposition 4.15. *Let (K, ν, Γ, k) be a finitely tamely ramified henselian valued field of mixed characteristic with perfect residue field. Let $e \geq 2$ be the absolute ramification index of (K, ν). Let T be the theory of (K, ν).*

(1) *If $k^\times$ is e-divisible, then $\lambda(T) = 1$.*
(2) *If there is a prime divisor l of e such that $\zeta_{l^n} \in k^\times$ and $\zeta_{l^{n+1}} \notin k^\times$ for some n, and Γ is a $\mathbb{Z}$-group, then $\lambda(T) = e+1$.*

Proof. (1) Suppose $k^\times$ is e-divisible. Let (K',ν',Γ',k') be a henselian valued field of mixed characteristic with a perfect residue field having absolute ramification index e. Suppose $k\equiv k'$ and $\Gamma\equiv\Gamma'$. By the proof of Proposition 4.5, we may assume that $k\cong k'$, $\Gamma\cong\Gamma'$, and both K and K' are $\aleph_1$-saturated. Consider the core fields $(K^\circ,\nu^\circ,k^\circ)$ and $((K')^\circ,(\nu')^\circ,(k')^\circ)$ of (K,ν) and (K',ν') respectively. Since $k^\times$ is e-divisible, so is $(k^\circ)^\times$. Then by Lemma 4.14, $(K^\circ,\nu^\circ)\cong((K')^\circ,(\nu')^\circ)$. By the proof of Theorem 4.7, we have $(K,\nu)\equiv(K',\nu')$. Thus $\lambda(T)=1$.

(2) Suppose there is a prime divisor l of e and a natural number n such that $\zeta_{l^n}\in k^\times$ and $\zeta_{l^{n+1}}\notin k^\times$, and $\Gamma\equiv\mathbb{Z}$. Let p be the characteristic of k and e be the absolute ramification index of (K,ν). Let $T_{p,e}$ be the theory introduced in the proof of Corollary 4.9. Set $T_0=T_{p,e}\cup\mathrm{Th}(\Gamma)\cup\mathrm{Th}(R_e)$. Consider the following theories:

- $T_1=T_0\cup\{\exists x(x^e-p=0)\}$;
- $T_2=T_0\cup\big\{\exists xy\big((x^e-py=0)\wedge\Phi_{l^n}(y)=0\big)\big\}$;
- $T_3=T_0\cup\big\{\neg\exists x(x^e-p=0),\neg\exists xy\big((x^e-py=0)\wedge\Phi_{l^n}(y)=0\big)\big\}$,

where $\Phi_{l^n}(X)\in\mathbb{Z}[X]$ is the l^n-th cyclotomic polynomial. By the proof of Lemma 3.12, we have

- each pairwise union of T_1, T_2, and T_3 is inconsistent;
- T_1 and T_2 are consistent;

and for a finitely tamely ramified henselian valued field (K',ν',Γ',k') of mixed characteristic $(0,p)$ having absolute ramification index e, if $k'\equiv k$, $\Gamma'\equiv\Gamma$, and $R_e'\equiv R_e$ for the e-th residue ring R_e' of (K',ν'), then there is $i\in\{1,2,3\}$

- $(K',\nu')\models T_i$.

Since $(K,\nu)\models T_0$ and there are at least two different complete theories containing T_0, we have $\lambda(T)\geq e+1$. By Corollary 4.13, we conclude that $\lambda(T)=e+1$.

□

For some wild cases, we have a lower bound for $\lambda(T)$.

Proposition 4.16. *Let p be a prime number and e be a positive integer divided by p. Let (K,ν,Γ,k) be a finitely ramified henselian valued field of mixed characteristic $(0,p)$ having absolute ramification index $e\geq 2$. Suppose k is perfect and Γ is $\mathbb{Z}$-group. Then $\lambda(T)\geq e+1$ for $T=\mathrm{Th}(K,\nu)$.*

Proof. The proof is similar to the proof of Proposition 4.15. Let $T_{p,e}$ and T_0 be the theory introduced in the proof of Proposition 4.15. We write $e=sp^r$ for positive integers s and r where s is prime to p. Let $\alpha\in\mathbb{Q}^{alg}$ be in the proof of Lemma 3.15 such that α is a uniformizer of M_r corresponding to the place above p where $M_r=\mathbb{Q}(\alpha)$ is the r-th subfield of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of degree p^r over $\mathbb{Q}$. Let $f(X)$ be the minimal polynomial of α over $\mathbb{Q}$. Consider the following theories:

- $T_1=T_0\cup\{\exists x(x^e-p=0)\}$;
- $T_2=T_0\cup\{\exists x(x^s-p=0),\ \exists x(f(x)=0)\}$;
- $T_3=T_0\cup\big\{\neg\exists x(x^e-p=0),\neg\big(\exists x(x^s-p=0)\wedge\exists x(f(x)=0)\big)\big\}$

By the proof of Lemma 3.15, we have

- each pairwise union of T_1, T_2, and T_3 is not consistent;
- T_1 and T_2 are consistent;

and for a ramified henselian valued field (K', ν', Γ', k') of mixed characteristic $(0, p)$ having absolute ramification index e, if $k' \equiv k$, $\Gamma' \equiv \Gamma$, and $R'_e \equiv R_e$ for the e-th residue ring R'_e of (K', ν'), then there is $i \in \{1, 2, 3\}$

- $(K', \nu') \models T_i$.

Since $(K, \nu) \models T_0$ and there are at least two different complete theories containing T_0, we have $\lambda(T) \geq e + 1$. □

We list some special cases of Proposition 4.15 and Proposition 4.16. For a positive integer s, we say that s^∞ divides $[k : \mathbb{F}_p]$ if there is a subfield k_n of k such that $[k_n : \mathbb{F}_p]$ is finite and s^n divides $[k_n : \mathbb{F}_p]$ for each $n \geq 1$.

Corollary 4.17. *Suppose (K, ν, Γ, k) is a finitely ramified henselian valued field of mixed characteristic $(0, p)$ having absolute ramification index $e \geq 2$. Let T be the theory of K. Let s be the order of the group $\mu_e \cap k^\times$ where μ_e is the group generated by ζ_e.*

Case $p \not| e$.

- *$\lambda(T) = 1$ when $k = k^{alg}$;*
- *$\lambda(T) = 1$ when K is a subfield of $\mathbb{C}_p$ and s^∞ divides $[k : \mathbb{F}_p]$;*
- *$\lambda(T) = e + 1$ when K is a subfield of $\mathbb{C}_p$ and s^∞ does not divide $[k : \mathbb{F}_p]$.*

Case $p | e$.

- *$\lambda(T) \geq e + 1$ when K is a subfield of $\mathbb{C}_p$.*

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Department of Mathematics, Yonsei University, 134 Sinchon-Dong, Seodaemun-Gu, Seoul 120-749, South Korea
e-mails: ljw@yonsei.ac.kr, wannim@yonsei.ac.kr